3.6 Quadratic Inequalities

Essential Question How can you solve a quadratic inequality?

EXPLORATION 1 Solving a Quadratic Inequality

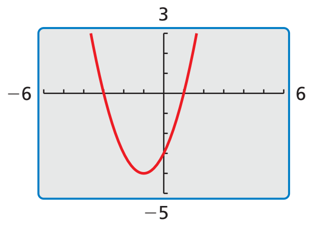
Work with a partner. The graphing calculator screen shows the graph of

$$f(x) = x^2 + 2x - 3.$$

Explain how you can use the graph to solve the inequality

$$x^2 + 2x - 3 \le 0.$$

Then solve the inequality.



USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore your understanding of concepts.

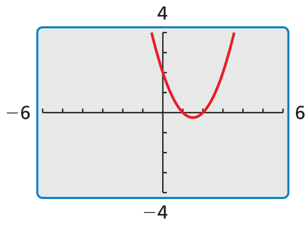
EXPLORATION 2 Solving Quadratic Inequalities

Work with a partner. Match each inequality with the graph of its related quadratic function. Then use the graph to solve the inequality.

a. $x^2 - 3x + 2 > 0$ **b.** $x^2 - 4x + 3 \le 0$ **c.** $x^2 - 2x - 3 < 0$

d. $x^2 + x - 2 \ge 0$ **e.** $x^2 - x - 2 < 0$ **f.** $x^2 - 4 > 0$

A.



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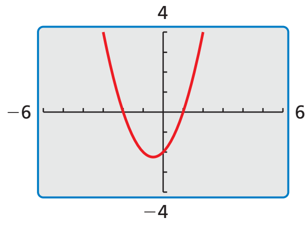

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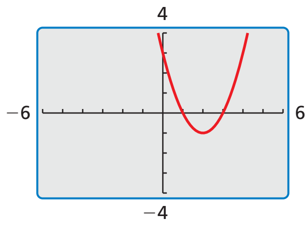

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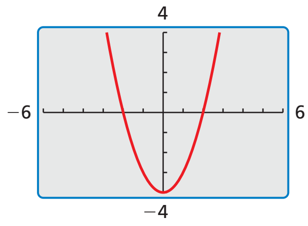

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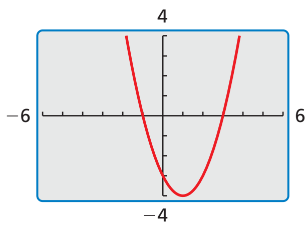

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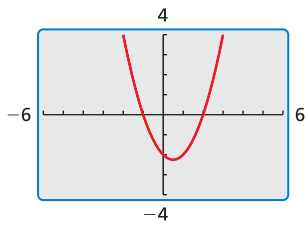


Communicate Your Answer

3. How can you solve a quadratic inequality?

4. Explain how you can use the graph in Exploration 1 to solve each inequality. Then solve each inequality.

a. $x^2 + 2x - 3 > 0$ **b.** $x^2 + 2x - 3 < 0$ **c.** $x^2 + 2x - 3 \ge 0$

3.6 Lesson

Core Vocabulary

quadratic inequality in two variables, *p. 140*
quadratic inequality in one variable, *p. 142*

Previous
linear inequality in two variables

What You Will Learn

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in one variable.

Graphing Quadratic Inequalities in Two Variables

A **quadratic inequality in two variables** can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$$y < ax^2 + bx + c \qquad y > ax^2 + bx + c$$

$$y \leq ax^2 + bx + c \qquad y \geq ax^2 + bx + c$$

The graph of any such inequality consists of all solutions (x, y) of the inequality.

Previously, you graphed linear inequalities in two variables. You can use a similar procedure to graph quadratic inequalities in two variables.

Core Concept

Graphing a Quadratic Inequality in Two Variables

To graph a quadratic inequality in one of the forms above, follow these steps.

Step 1 Graph the parabola with the equation $y = ax^2 + bx + c$. Make the parabola *dashed* for inequalities with $<$ or $>$ and *solid* for inequalities with $\leq$ or $\geq$.

Step 2 Test a point (x, y) inside the parabola to determine whether the point is a solution of the inequality.

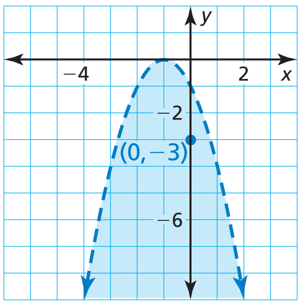
Step 3 Shade the region inside the parabola if the point from Step 2 is a solution. Shade the region outside the parabola if it is not a solution.

EXAMPLE 1 Graphing a Quadratic Inequality in Two Variables

Graph $y < -x^2 - 2x - 1$.

SOLUTION

Step 1 Graph $y = -x^2 - 2x - 1$. Because the inequality symbol is $<$, make the parabola dashed.



Step 2 Test a point inside the parabola, such as $(0, -3)$.

$$y < -x^2 - 2x - 1$$

$$-3 \stackrel{?}{<} -0^2 - 2(0) - 1$$

$$-3 < -1 \checkmark$$

So, $(0, -3)$ is a solution of the inequality.

Step 3 Shade the region inside the parabola.

LOOKING FOR STRUCTURE

Notice that testing a point is less complicated when the x-value is 0 (the point is on the y-axis).

EXAMPLE 2 Using a Quadratic Inequality in Real Life

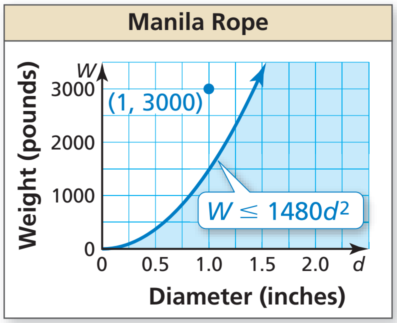
A manila rope used for rappelling down a cliff can safely support a weight W (in pounds) provided

$$W \leq 1480d^2$$

where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

SOLUTION

Graph $W = 1480d^2$ for nonnegative values of d. Because the inequality symbol is ≤, make the parabola solid. Test a point inside the parabola, such as (1, 3000).



$$W \leq 1480d^2$$

$$3000 \overset{?}{\leq} 1480(1)^2$$

$$3000 \not\leq 1480$$

▶ Because (1, 3000) is not a solution, shade the region outside the parabola. The shaded region represents weights that can be supported by ropes with various diameters.

Graphing a *system* of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the system. Then identify the region in the coordinate plane common to all of the graphs. This region is called the *graph of the system*.

EXAMPLE 3 Graphing a System of Quadratic Inequalities

Graph the system of quadratic inequalities.

$y < -x^2 + 3$ Inequality 1

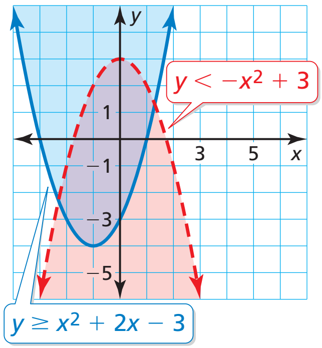
$y \geq x^2 + 2x - 3$ Inequality 2

SOLUTION

Step 1 Graph $y < -x^2 + 3$. The graph is the red region inside (but not including) the parabola $y = -x^2 + 3$.

Step 2 Graph $y \geq x^2 + 2x - 3$. The graph is the blue region inside and including the parabola $y = x^2 + 2x - 3$.

Step 3 Identify the purple region where the two graphs overlap. This region is the graph of the system.



Check

Check that a point in the solution region, such as (0, 0), is a solution of the system.

$$y < -x^2 + 3$$

$$0 \overset{?}{<} -0^2 + 3$$

$$0 < 3 \checkmark$$

$$y \geq x^2 + 2x - 3$$

$$0 \overset{?}{\geq} 0^2 + 2(0) - 3$$

$$0 \geq -3 \checkmark$$

Monitoring Progress

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Graph the inequality.

1. $y \geq x^2 + 2x - 8$ **2.** $y \leq 2x^2 - x - 1$ **3.** $y > -x^2 + 2x + 4$

4. Graph the system of inequalities consisting of $y \leq -x^2$ and $y > x^2 - 3$.

Solving Quadratic Inequalities in One Variable

A **quadratic inequality in one variable** can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$ax^2 + bx + c < 0 \qquad ax^2 + bx + c > 0 \qquad ax^2 + bx + c \leq 0 \qquad ax^2 + bx + c \geq 0$

You can solve quadratic inequalities using algebraic methods or graphs.

EXAMPLE 4 Solving a Quadratic Inequality Algebraically

Solve $x^2 - 3x - 4 < 0$ algebraically.

SOLUTION

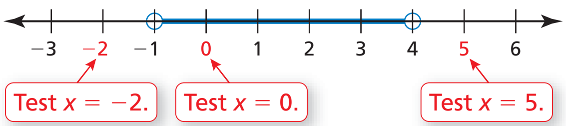
First, write and solve the equation obtained by replacing $<$ with $=$.

$x^2 - 3x - 4 = 0$ Write the related equation.

$(x - 4)(x + 1) = 0$ Factor.

$x = 4$ or $x = -1$ Zero-Product Property

The numbers -1 and 4 are the *critical values* of the original inequality. Plot -1 and 4 on a number line, using open dots because the values do not satisfy the inequality. The critical x-values partition the number line into three intervals. Test an x-value in each interval to determine whether it satisfies the inequality.



$(-2)^2 - 3(-2) - 4 = 6 \nless 0 \qquad 0^2 - 3(0) - 4 = -4 < 0$ ✓ $\qquad 5^2 - 3(5) - 4 = 6 \nless 0$

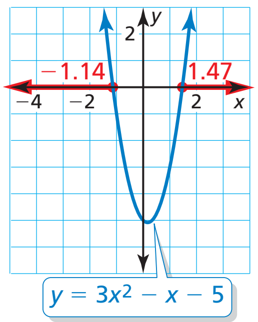
▶ So, the solution is $-1 < x < 4$.

Another way to solve $ax^2 + bx + c < 0$ is to first graph the related function $y = ax^2 + bx + c$. Then, because the inequality symbol is $<$, identify the x-values for which the graph lies *below* the x-axis. You can use a similar procedure to solve quadratic inequalities that involve $\leq$, $>$, or $\geq$.

EXAMPLE 5 Solving a Quadratic Inequality by Graphing

Solve $3x^2 - x - 5 \geq 0$ by graphing.

SOLUTION



The solution consists of the x-values for which the graph of $y = 3x^2 - x - 5$ lies on or above the x-axis. Find the x-intercepts of the graph by letting $y = 0$ and using the Quadratic Formula to solve $0 = 3x^2 - x - 5$ for x.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)} \qquad a = 3, b = -1, c = -5$$

$$x = \frac{1 \pm \sqrt{61}}{6} \qquad \text{Simplify.}$$

The solutions are $x \approx -1.14$ and $x \approx 1.47$. Sketch a parabola that opens up and has -1.14 and 1.47 as x-intercepts. The graph lies on or above the x-axis to the left of (and including) $x = -1.14$ and to the right of (and including) $x = 1.47$.

▶ The solution of the inequality is approximately $x \leq -1.14$ or $x \geq 1.47$.

EXAMPLE 6 Modeling with Mathematics

A rectangular parking lot must have a perimeter of 440 feet and an area of at least 8000 square feet. Describe the possible lengths of the parking lot.

SOLUTION

1. **Understand the Problem** You are given the perimeter and the minimum area of a parking lot. You are asked to determine the possible lengths of the parking lot.
2. **Make a Plan** Use the perimeter and area formulas to write a quadratic inequality describing the possible lengths of the parking lot. Then solve the inequality.
3. **Solve the Problem** Let ℓ represent the length (in feet) and let w represent the width (in feet) of the parking lot.

Perimeter $= 440$	Area ≥ 8000
$2\ell + 2w = 440$	$\ell w \geq 8000$

Solve the perimeter equation for w to obtain $w = 220 - \ell$. Substitute this into the area inequality to obtain a quadratic inequality in one variable.

$\ell w \geq 8000$ Write the area inequality.

$\ell(220 - \ell) \geq 8000$ Substitute $220 - \ell$ for w.

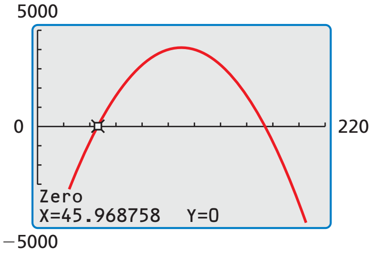
$220\ell - \ell^2 \geq 8000$ Distributive Property

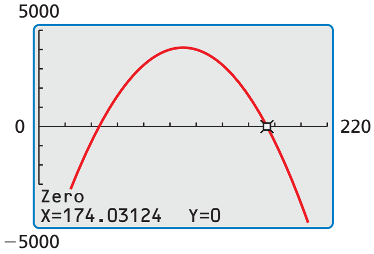
$-\ell^2 + 220\ell - 8000 \geq 0$ Write in standard form.

ANOTHER WAY

You can graph each side of $220\ell - \ell^2 = 8000$ and use the intersection points to determine when $220\ell - \ell^2$ is greater than or equal to 8000.

Use a graphing calculator to find the ℓ-intercepts of $y = -\ell^2 + 220\ell - 8000$.





USING TECHNOLOGY

Variables displayed when using technology may not match the variables used in applications. In the graphs shown, the length ℓ corresponds to the independent variable *x*.

The ℓ-intercepts are $\ell \approx 45.97$ and $\ell \approx 174.03$. The solution consists of the ℓ-values for which the graph lies on or above the ℓ-axis. The graph lies on or above the ℓ-axis when $45.97 \leq \ell \leq 174.03$.

▶ So, the approximate length of the parking lot is at least 46 feet and at most 174 feet.

4. **Look Back** Choose a length in the solution region, such as $\ell = 100$, and find the width. Then check that the dimensions satisfy the original area inequality.

$2\ell + 2w = 440$	$\ell w \geq 8000$
$2(100) + 2w = 440$	$100(120) \overset{?}{\geq} 8000$
$w = 120$	$12{,}000 \geq 8000$ ✓

Monitoring Progress

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Solve the inequality.

5. $2x^2 + 3x \leq 2$
6. $-3x^2 - 4x + 1 < 0$
7. $2x^2 + 2 > -5x$
8. **WHAT IF?** In Example 6, the area must be at least 8500 square feet. Describe the possible lengths of the parking lot.

3.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. WRITING Compare the graph of a quadratic inequality in one variable to the graph of a quadratic inequality in two variables.

2. WRITING Explain how to solve $x^2 + 6x - 8 < 0$ using algebraic methods and using graphs.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the inequality with its graph. Explain your reasoning.

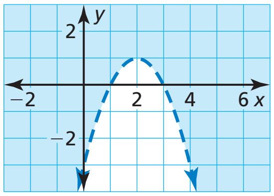
3. $y \le x^2 + 4x + 3$

4. $y > -x^2 + 4x - 3$

5. $y < x^2 - 4x + 3$

6. $y \ge x^2 + 4x + 3$

A.



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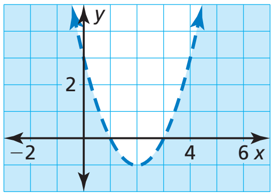

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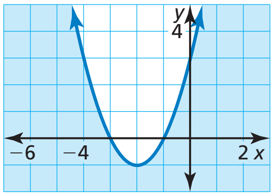

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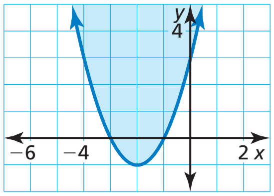


In Exercises 7–14, graph the inequality. *(See Example 1.)*

7. $y < -x^2$

8. $y \ge 4x^2$

9. $y > x^2 - 9$

10. $y < x^2 + 5$

11. $y \le x^2 + 5x$

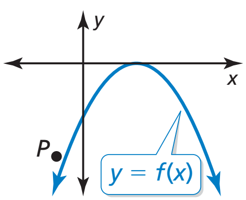
12. $y \ge -2x^2 + 9x - 4$

13. $y > 2(x + 3)^2 - 1$

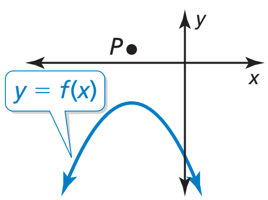
14. $y \le \left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$

ANALYZING RELATIONSHIPS In Exercises 15 and 16, use the graph to write an inequality in terms of $f(x)$ so point P is a solution.

15.

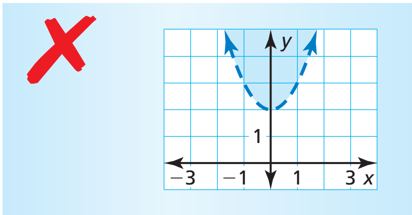


16.



ERROR ANALYSIS In Exercises 17 and 18, describe and correct the error in graphing $y \ge x^2 + 2$.

17.



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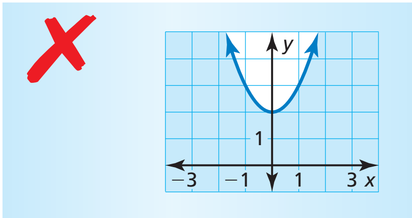


19. MODELING WITH MATHEMATICS A hardwood shelf in a wooden bookcase can safely support a weight W (in pounds) provided $W \le 115x^2$, where x is the thickness (in inches) of the shelf. Graph the inequality and interpret the solution. *(See Example 2.)*

20. MODELING WITH MATHEMATICS A wire rope can safely support a weight W (in pounds) provided $W \le 8000d^2$, where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

In Exercises 21–26, graph the system of quadratic inequalities. *(See Example 3.)*

21. $y \ge 2x^2$
$y < -x^2 + 1$

22. $y > -5x^2$
$y > 3x^2 - 2$

23. $y \le -x^2 + 4x - 4$
$y < x^2 + 2x - 8$

24. $y \ge x^2 - 4$
$y \le -2x^2 + 7x + 4$

25. $y \ge 2x^2 + x - 5$
$y < -x^2 + 5x + 10$

26. $y \ge x^2 - 3x - 6$
$y \ge x^2 + 7x + 6$

In Exercises 27–34, solve the inequality algebraically. *(See Example 4.)*

27. $4x^2 < 25$

28. $x^2 + 10x + 9 < 0$

29. $x^2 - 11x \geq -28$

30. $3x^2 - 13x > -10$

31. $2x^2 - 5x - 3 \leq 0$

32. $4x^2 + 8x - 21 \geq 0$

33. $\frac{1}{2}x^2 - x > 4$

34. $-\frac{1}{2}x^2 + 4x \leq 1$

In Exercises 35–42, solve the inequality by graphing. *(See Example 5.)*

35. $x^2 - 3x + 1 < 0$

36. $x^2 - 4x + 2 > 0$

37. $x^2 + 8x > -7$

38. $x^2 + 6x < -3$

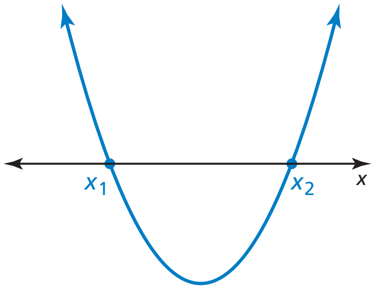
39. $3x^2 - 8 \leq -2x$

40. $3x^2 + 5x - 3 < 1$

41. $\frac{1}{3}x^2 + 2x \geq 2$

42. $\frac{3}{4}x^2 + 4x \geq 3$

43. **DRAWING CONCLUSIONS** Consider the graph of the function $f(x) = ax^2 + bx + c$.



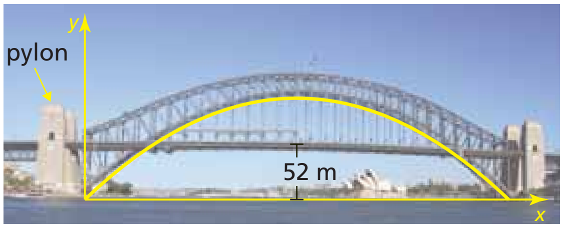
a. What are the solutions of $ax^2 + bx + c < 0$?

b. What are the solutions of $ax^2 + bx + c > 0$?

c. The graph of g represents a reflection in the x-axis of the graph of f. For which values of x is $g(x)$ positive?

44. **MODELING WITH MATHEMATICS** A rectangular fountain display has a perimeter of 400 feet and an area of at least 9100 feet. Describe the possible widths of the fountain. *(See Example 6.)*

45. **MODELING WITH MATHEMATICS** The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by $y = -0.00211x^2 + 1.06x$, where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. For what distances x is the arch above the road?



46. **PROBLEM SOLVING** The number T of teams that have participated in a robot-building competition for high-school students over a recent period of time x (in years) can be modeled by

$$T(x) = 17.155x^2 + 193.68x + 235.81,\ 0 \leq x \leq 6.$$

After how many years is the number of teams greater than 1000? Justify your answer.

47. **PROBLEM SOLVING** A study found that a driver's reaction time $A(x)$ to audio stimuli and his or her reaction time $V(x)$ to visual stimuli (both in milliseconds) can be modeled by

$$A(x) = 0.0051x^2 - 0.319x + 15,\ 16 \leq x \leq 70$$

$$V(x) = 0.005x^2 - 0.23x + 22,\ 16 \leq x \leq 70$$

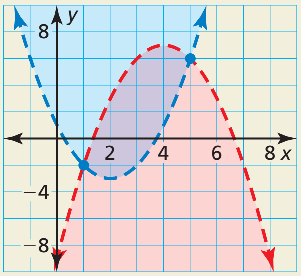
where x is the age (in years) of the driver.

a. Write an inequality that you can use to find the x-values for which $A(x)$ is less than $V(x)$.

b. Use a graphing calculator to solve the inequality $A(x) < V(x)$. Describe how you used the domain $16 \leq x \leq 70$ to determine a reasonable solution.

c. Based on your results from parts (a) and (b), do you think a driver would react more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.

48. HOW DO YOU SEE IT? The graph shows a system of quadratic inequalities.



a. Identify two solutions of the system.

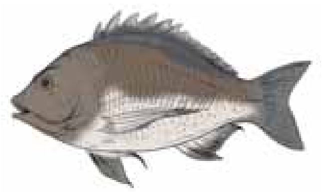
b. Are the points (1, −2) and (5, 6) solutions of the system? Explain.

c. Is it possible to change the inequality symbol(s) so that one, but not both of the points in part (b), is a solution of the system? Explain.

49. MODELING WITH MATHEMATICS The length L (in millimeters) of the larvae of the black porgy fish can be modeled by

$$L(x) = 0.00170x^2 + 0.145x + 2.35,\ 0 \le x \le 40$$

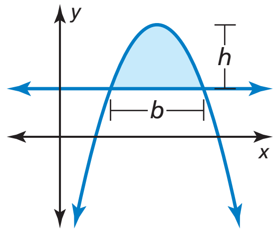
where x is the age (in days) of the larvae. Write and solve an inequality to find at what ages a larva's length tends to be greater than 10 millimeters. Explain how the given domain affects the solution.



50. MAKING AN ARGUMENT You claim the system of inequalities below, where a and b are real numbers, has no solution. Your friend claims the system will always have at least one solution. Who is correct? Explain.

$$y < (x + a)^2$$
$$y < (x + b)^2$$

51. MATHEMATICAL CONNECTIONS The area A of the region bounded by a parabola and a horizontal line can be modeled by $A = \frac{2}{3}bh$, where b and h are as defined in the diagram. Find the area of the region determined by each pair of inequalities.



a. $y \le -x^2 + 4x$
$y \ge 0$

b. $y \ge x^2 - 4x - 5$
$y \le 7$

52. THOUGHT PROVOKING Draw a company logo that is created by the intersection of two quadratic inequalities. Justify your answer.

53. REASONING A truck that is 11 feet tall and 7 feet wide is traveling under an arch. The arch can be modeled by $y = -0.0625x^2 + 1.25x + 5.75$, where x and y are measured in feet.

a. Will the truck fit under the arch? Explain.

b. What is the maximum width that a truck 11 feet tall can have and still make it under the arch?

c. What is the maximum height that a truck 7 feet wide can have and still make it under the arch?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the function. Label the x-intercept(s) and the y-intercept. *(Section 2.2)*

54. $f(x) = (x + 7)(x - 9)$ **55.** $g(x) = (x - 2)^2 - 4$ **56.** $h(x) = -x^2 + 5x - 6$

Find the minimum value or maximum value of the function. Then describe where the function is increasing and decreasing. *(Section 2.2)*

57. $f(x) = -x^2 - 6x - 10$ **58.** $h(x) = \frac{1}{2}(x + 2)^2 - 1$

59. $f(x) = -(x - 3)(x + 7)$ **60.** $h(x) = x^2 + 3x - 18$