3.4 Using the Quadratic Formula

Essential Question How can you derive a general formula for solving a quadratic equation?

EXPLORATION 1 Deriving the Quadratic Formula

Work with a partner. Analyze and describe what is done in each step in the development of the Quadratic Formula.

Step	Justification
$ax^2 + bx + c = 0$	
$ax^2 + bx = -c$	
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$	
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2\lvert a \rvert}$	
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	

The result is the Quadratic Formula.

REASONING ABSTRACTLY

To be proficient in math, you need to create a coherent representation of the problem at hand.

EXPLORATION 2 Using the Quadratic Formula

Work with a partner. Use the Quadratic Formula to solve each equation.

a. $x^2 - 4x + 3 = 0$

b. $x^2 - 2x + 2 = 0$

c. $x^2 + 2x - 3 = 0$

d. $x^2 + 4x + 4 = 0$

e. $x^2 - 6x + 10 = 0$

f. $x^2 + 4x + 6 = 0$

Communicate Your Answer

3. How can you derive a general formula for solving a quadratic equation?

4. Summarize the following methods you have learned for solving quadratic equations: graphing, using square roots, factoring, completing the square, and using the Quadratic Formula.

3.4 Lesson

Core Vocabulary

Quadratic Formula, *p. 122*
discriminant, *p. 124*

What You Will Learn

- Solve quadratic equations using the Quadratic Formula.
- Analyze the discriminant to determine the number and type of solutions.
- Solve real-life problems.

Solving Equations Using the Quadratic Formula

Previously, you solved quadratic equations by completing the square. In the Exploration, you developed a formula that gives the solutions of any quadratic equation by completing the square once for the general equation $ax^2 + bx + c = 0$. The formula for the solutions is called the **Quadratic Formula**.

Core Concept

The Quadratic Formula

Let a, b, and c be real numbers such that $a \neq 0$. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

COMMON ERROR

Remember to write the quadratic equation in standard form before applying the Quadratic Formula.

EXAMPLE 1 Solving an Equation with Two Real Solutions

Solve $x^2 + 3x = 5$ using the Quadratic Formula.

SOLUTION

$x^2 + 3x = 5$ — Write original equation.

$x^2 + 3x - 5 = 0$ — Write in standard form.

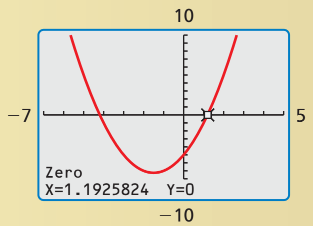
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Quadratic Formula

$x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(-5)}}{2(1)}$ — Substitute 1 for *a*, 3 for *b*, and −5 for *c*.

$x = \dfrac{-3 \pm \sqrt{29}}{2}$ — Simplify.

So, the solutions are $x = \dfrac{-3 + \sqrt{29}}{2} \approx 1.19$ and $x = \dfrac{-3 - \sqrt{29}}{2} \approx -4.19$.

Check Graph $y = x^2 + 3x - 5$. The x-intercepts are about -4.19 and about 1.19. ✓



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Solve the equation using the Quadratic Formula.

1. $x^2 - 6x + 4 = 0$ **2.** $2x^2 + 4 = -7x$ **3.** $5x^2 = x + 8$

EXAMPLE 2 Solving an Equation with One Real Solution

Solve $25x^2 - 8x = 12x - 4$ using the Quadratic Formula.

SOLUTION

$$25x^2 - 8x = 12x - 4 \qquad \text{Write original equation.}$$

$$25x^2 - 20x + 4 = 0 \qquad \text{Write in standard form.}$$

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(25)(4)}}{2(25)} \qquad a = 25,\ b = -20,\ c = 4$$

$$x = \frac{20 \pm \sqrt{0}}{50} \qquad \text{Simplify.}$$

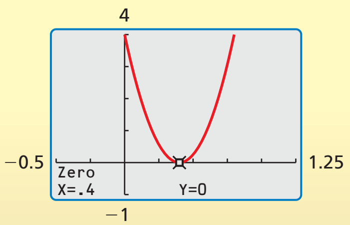
$$x = \frac{2}{5} \qquad \text{Simplify.}$$

▶ So, the solution is $x = \frac{2}{5}$. You can check this by graphing $y = 25x^2 - 20x + 4$. The only x-intercept is $\frac{2}{5}$.

ANOTHER WAY

You can also use factoring to solve $25x^2 - 20x + 4 = 0$ because the left side factors as $(5x - 2)^2$.

Check



EXAMPLE 3 Solving an Equation with Imaginary Solutions

Solve $-x^2 + 4x = 13$ using the Quadratic Formula.

SOLUTION

$$-x^2 + 4x = 13 \qquad \text{Write original equation.}$$

$$-x^2 + 4x - 13 = 0 \qquad \text{Write in standard form.}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(-13)}}{2(-1)} \qquad a = -1,\ b = 4,\ c = -13$$

$$x = \frac{-4 \pm \sqrt{-36}}{-2} \qquad \text{Simplify.}$$

$$x = \frac{-4 \pm 6i}{-2} \qquad \text{Write in terms of } i.$$

$$x = 2 \pm 3i \qquad \text{Simplify.}$$

▶ The solutions are $x = 2 + 3i$ and $x = 2 - 3i$.

COMMON ERROR

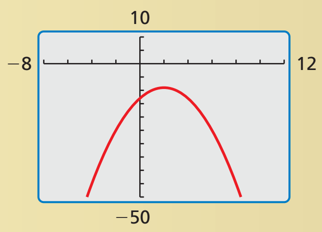
Remember to divide the real part *and* the imaginary part by -2 when simplifying.

Check Graph $y = -x^2 + 4x - 13$. There are no x-intercepts. So, the original equation has no real solutions. The algebraic check for one of the imaginary solutions is shown.

$$-(2 + 3i)^2 + 4(2 + 3i) \stackrel{?}{=} 13$$

$$5 - 12i + 8 + 12i \stackrel{?}{=} 13$$

$$13 = 13 \ ✓$$



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Solve the equation using the Quadratic Formula.

4. $x^2 + 41 = -8x$ **5.** $-9x^2 = 30x + 25$ **6.** $5x - 7x^2 = 3x + 4$

Analyzing the Discriminant

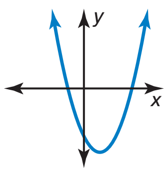
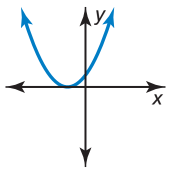
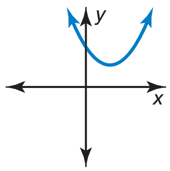
In the Quadratic Formula, the expression $b^2 - 4ac$ is called the **discriminant** of the associated equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ discriminant

You can analyze the discriminant of a quadratic equation to determine the number and type of solutions of the equation.

Core Concept

Analyzing the Discriminant of $ax^2 + bx + c = 0$

Value of discriminant	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Number and type of solutions	Two real solutions	One real solution	Two imaginary solutions
Graph of $y = ax^2 + bx + c$	 Two x-intercepts	 One x-intercept	 No x-intercept

EXAMPLE 4 Analyzing the Discriminant

Find the discriminant of the quadratic equation and describe the number and type of solutions of the equation.

a. $x^2 - 6x + 10 = 0$ **b.** $x^2 - 6x + 9 = 0$ **c.** $x^2 - 6x + 8 = 0$

SOLUTION

Equation	**Discriminant**	**Solution(s)**
$\mathbf{ax^2 + bx + c = 0}$	$\mathbf{b^2 - 4ac}$	$\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$
a. $x^2 - 6x + 10 = 0$	$(-6)^2 - 4(1)(10) = -4$	Two imaginary: $3 \pm i$
b. $x^2 - 6x + 9 = 0$	$(-6)^2 - 4(1)(9) = 0$	One real: 3
c. $x^2 - 6x + 8 = 0$	$(-6)^2 - 4(1)(8) = 4$	Two real: 2, 4

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Find the discriminant of the quadratic equation and describe the number and type of solutions of the equation.

7. $4x^2 + 8x + 4 = 0$

8. $\frac{1}{2}x^2 + x - 1 = 0$

9. $5x^2 = 8x - 13$

10. $7x^2 - 3x = 6$

11. $4x^2 + 6x = -9$

12. $-5x^2 + 1 = 6 - 10x$

EXAMPLE 5 Writing an Equation

Find a possible pair of integer values for a and c so that the equation $ax^2 - 4x + c = 0$ has one real solution. Then write the equation.

SOLUTION

In order for the equation to have one real solution, the discriminant must equal 0.

$$b^2 - 4ac = 0 \quad \text{Write the discriminant.}$$

$$(-4)^2 - 4ac = 0 \quad \text{Substitute } -4 \text{ for } b.$$

$$16 - 4ac = 0 \quad \text{Evaluate the power.}$$

$$-4ac = -16 \quad \text{Subtract 16 from each side.}$$

$$ac = 4 \quad \text{Divide each side by } -4.$$

Because $ac = 4$, choose two integers whose product is 4, such as $a = 1$ and $c = 4$.

▶ So, one possible equation is $x^2 - 4x + 4 = 0$.

ANOTHER WAY

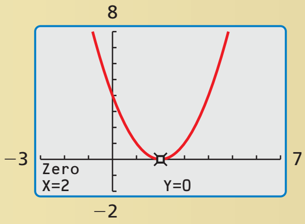
Another possible equation in Example 5 is $4x^2 - 4x + 1 = 0$. You can obtain this equation by letting $a = 4$ and $c = 1$.

Check Graph $y = x^2 - 4x + 4$. The only x-intercept is 2. You can also check by factoring.

$$x^2 - 4x + 4 = 0$$

$$(x - 2)^2 = 0$$

$$x = 2 \checkmark$$



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13. Find a possible pair of integer values for a and c so that the equation $ax^2 + 3x + c = 0$ has two real solutions. Then write the equation.

The table shows five methods for solving quadratic equations. For a given equation, it may be more efficient to use one method instead of another. Suggestions about when to use each method are shown below.

Concept Summary

Methods for Solving Quadratic Equations

Method	When to Use
Graphing	Use when approximate solutions are adequate.
Using square roots	Use when solving an equation that can be written in the form $u^2 = d$, where u is an algebraic expression.
Factoring	Use when a quadratic equation can be factored easily.
Completing the square	Can be used for *any* quadratic equation $ax^2 + bx + c = 0$ but is simplest to apply when $a = 1$ and b is an even number.
Quadratic Formula	Can be used for *any* quadratic equation.

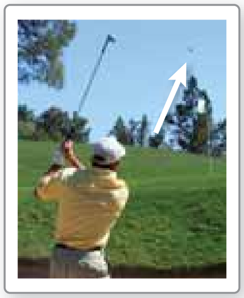
Solving Real-Life Problems

The function $h = -16t^2 + h_0$ is used to model the height of a *dropped* object. For an object that is *launched* or *thrown*, an extra term v_0t must be added to the model to account for the object's initial vertical velocity v_0 (in feet per second). Recall that h is the height (in feet), t is the time in motion (in seconds), and h_0 is the initial height (in feet).

$h = -16t^2 + h_0$ — Object is dropped.

$h = -16t^2 + v_0t + h_0$ — Object is launched or thrown.

As shown below, the value of v_0 can be positive, negative, or zero depending on whether the object is launched upward, downward, or parallel to the ground.



$V_0 > 0$

$V_0 < 0$

$V_0 = 0$

EXAMPLE 6 Modeling a Launched Object

A juggler tosses a ball into the air. The ball leaves the juggler's hand 4 feet above the ground and has an initial vertical velocity of 30 feet per second. The juggler catches the ball when it falls back to a height of 3 feet. How long is the ball in the air?

SOLUTION

Because the ball is *thrown*, use the model $h = -16t^2 + v_0t + h_0$. To find how long the ball is in the air, solve for t when $h = 3$.

$h = -16t^2 + v_0t + h_0$ — Write the height model.

$3 = -16t^2 + 30t + 4$ — Substitute 3 for h, 30 for v_0, and 4 for h_0.

$0 = -16t^2 + 30t + 1$ — Write in standard form.

This equation is not factorable, and completing the square would result in fractions. So, use the Quadratic Formula to solve the equation.

$t = \dfrac{-30 \pm \sqrt{30^2 - 4(-16)(1)}}{2(-16)}$ — $a = -16, b = 30, c = 1$

$t = \dfrac{-30 \pm \sqrt{964}}{-32}$ — Simplify.

$t \approx -0.033$ or $t \approx 1.9$ — Use a calculator.

▶ Reject the negative solution, -0.033, because the ball's time in the air cannot be negative. So, the ball is in the air for about 1.9 seconds.

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14. WHAT IF? The ball leaves the juggler's hand with an initial vertical velocity of 40 feet per second. How long is the ball in the air?

3.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** When a, b, and c are real numbers such that $a \neq 0$, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x =$ ____________.

2. **COMPLETE THE SENTENCE** You can use the ____________ of a quadratic equation to determine the number and type of solutions of the equation.

3. **WRITING** Describe the number and type of solutions when the value of the discriminant is negative.

4. **WRITING** Which two methods can you use to solve *any* quadratic equation? Explain when you might prefer to use one method over the other.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–18, solve the equation using the Quadratic Formula. Use a graphing calculator to check your solution(s). *(See Examples 1, 2, and 3.)*

5. $x^2 - 4x + 3 = 0$
6. $3x^2 + 6x + 3 = 0$
7. $x^2 + 6x + 15 = 0$
8. $6x^2 - 2x + 1 = 0$
9. $x^2 - 14x = -49$
10. $2x^2 + 4x = 30$
11. $3x^2 + 5 = -2x$
12. $-3x = 2x^2 - 4$
13. $-10x = -25 - x^2$
14. $-5x^2 - 6 = -4x$
15. $-4x^2 + 3x = -5$
16. $x^2 + 121 = -22x$
17. $-z^2 = -12z + 6$
18. $-7w + 6 = -4w^2$

In Exercises 19–26, find the discriminant of the quadratic equation and describe the number and type of solutions of the equation. *(See Example 4.)*

19. $x^2 + 12x + 36 = 0$
20. $x^2 - x + 6 = 0$
21. $4n^2 - 4n - 24 = 0$
22. $-x^2 + 2x + 12 = 0$
23. $4x^2 = 5x - 10$
24. $-18p = p^2 + 81$
25. $24x = -48 - 3x^2$
26. $-2x^2 - 6 = x$

27. **USING EQUATIONS** What are the complex solutions of the equation $2x^2 - 16x + 50 = 0$?

    (A) $4 + 3i, 4 - 3i$ (B) $4 + 12i, 4 - 12i$

    (C) $16 + 3i, 16 - 3i$ (D) $16 + 12i, 16 - 12i$

28. **USING EQUATIONS** Determine the number and type of solutions to the equation $x^2 + 7x = -11$.

    (A) two real solutions

    (B) one real solution

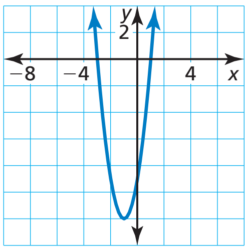
    (C) two imaginary solutions

    (D) one imaginary solution

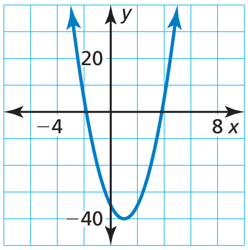
ANALYZING EQUATIONS **In Exercises 29–32, use the discriminant to match each quadratic equation with the correct graph of the related function. Explain your reasoning.**

29. $x^2 - 6x + 25 = 0$
30. $2x^2 - 20x + 50 = 0$
31. $3x^2 + 6x - 9 = 0$
32. $5x^2 - 10x - 35 = 0$

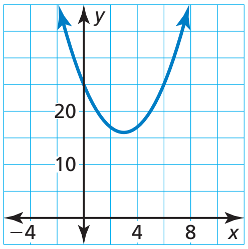
A.



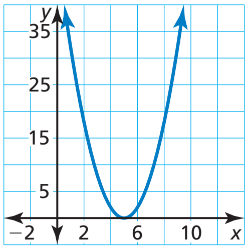
B.



C.



D.



ERROR ANALYSIS In Exercises 33 and 34, describe and correct the error in solving the equation.

33.

$$x^2 + 10x + 74 = 0$$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(74)}}{2(1)}$$

$$= \frac{-10 \pm \sqrt{-196}}{2}$$

$$= \frac{-10 \pm 14}{2}$$

$$= -12 \text{ or } 2$$

34.

$$x^2 + 6x + 8 = 2$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{4}}{2}$$

$$= \frac{-6 \pm 2}{2}$$

$$= -2 \text{ or } -4$$

OPEN-ENDED In Exercises 35–40, find a possible pair of integer values for a and c so that the quadratic equation has the given solution(s). Then write the equation. *(See Example 5.)*

35. $ax^2 + 4x + c = 0$; two imaginary solutions

36. $ax^2 + 6x + c = 0$; two real solutions

37. $ax^2 - 8x + c = 0$; two real solutions

38. $ax^2 - 6x + c = 0$; one real solution

39. $ax^2 + 10x = c$; one real solution

40. $-4x + c = -ax^2$; two imaginary solutions

USING STRUCTURE In Exercises 41–46, use the Quadratic Formula to write a quadratic equation that has the given solutions.

41. $x = \dfrac{-8 \pm \sqrt{-176}}{-10}$

42. $x = \dfrac{15 \pm \sqrt{-215}}{22}$

43. $x = \dfrac{-4 \pm \sqrt{-124}}{-14}$

44. $x = \dfrac{-9 \pm \sqrt{137}}{4}$

45. $x = \dfrac{-4 \pm 2}{6}$

46. $x = \dfrac{2 \pm 4}{-2}$

COMPARING METHODS In Exercises 47–58, solve the quadratic equation using the Quadratic Formula. Then solve the equation using another method. Which method do you prefer? Explain.

47. $3x^2 - 21 = 3$

48. $5x^2 + 38 = 3$

49. $2x^2 - 54 = 12x$

50. $x^2 = 3x + 15$

51. $x^2 - 7x + 12 = 0$

52. $x^2 + 8x - 13 = 0$

53. $5x^2 - 50x = -135$

54. $8x^2 + 4x + 5 = 0$

55. $-3 = 4x^2 + 9x$

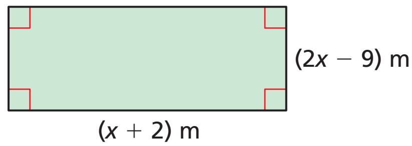
56. $-31x + 56 = -x^2$

57. $x^2 = 1 - x$

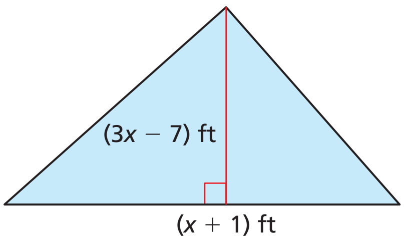
58. $9x^2 + 36x + 72 = 0$

MATHEMATICAL CONNECTIONS In Exercises 59 and 60, find the value for x.

59. Area of the rectangle $= 24\text{ m}^2$



60. Area of the triangle $= 8\text{ ft}^2$

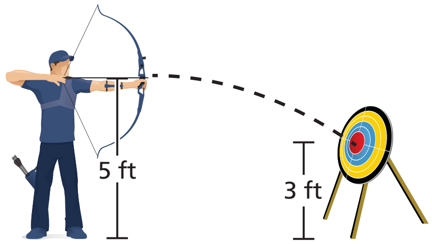


61. **MODELING WITH MATHEMATICS** A lacrosse player throws a ball in the air from an initial height of 7 feet. The ball has an initial vertical velocity of 90 feet per second. Another player catches the ball when it is 3 feet above the ground. How long is the ball in the air? *(See Example 6.)*

62. **NUMBER SENSE** Suppose the quadratic equation $ax^2 + 5x + c = 0$ has one real solution. Is it possible for a and c to be integers? rational numbers? Explain your reasoning. Then describe the possible values of a and c.

63. MODELING WITH MATHEMATICS In a volleyball game, a player on one team spikes the ball over the net when the ball is 10 feet above the court. The spike drives the ball downward with an initial vertical velocity of 55 feet per second. How much time does the opposing team have to return the ball before it touches the court?

64. MODELING WITH MATHEMATICS An archer is shooting at targets. The height of the arrow is 5 feet above the ground. Due to safety rules, the archer must aim the arrow parallel to the ground.



a. How long does it take for the arrow to hit a target that is 3 feet above the ground?

b. What method did you use to solve the quadratic equation? Explain.

65. PROBLEM SOLVING A rocketry club is launching model rockets. The launching pad is 30 feet above the ground. Your model rocket has an initial vertical velocity of 105 feet per second. Your friend's model rocket has an initial vertical velocity of 100 feet per second.

a. Use a graphing calculator to graph the equations of both model rockets. Compare the paths.

b. After how many seconds is your rocket 119 feet above the ground? Explain the reasonableness of your answer(s).

66. PROBLEM SOLVING The number A of tablet computers sold (in millions) can be modeled by the function $A = 4.5t^2 + 43.5t + 17$, where t represents the year after 2010.

a. In what year did the tablet computer sales reach 65 million?

b. Find the average rate of change from 2010 to 2012 and interpret the meaning in the context of the situation.

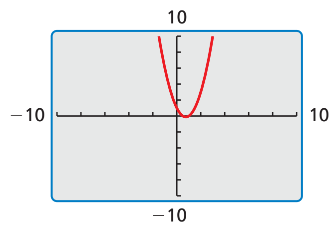
c. Do you think this model will be accurate after a new, innovative computer is developed? Explain.

67. MODELING WITH MATHEMATICS A gannet is a bird that feeds on fish by diving into the water. A gannet spots a fish on the surface of the water and dives 100 feet to catch it. The bird plunges toward the water with an initial vertical velocity of -88 feet per second.

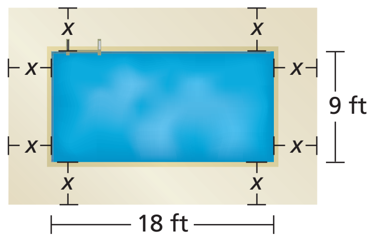
a. How much time does the fish have to swim away?

b. Another gannet spots the same fish, and it is only 84 feet above the water and has an initial vertical velocity of -70 feet per second. Which bird will reach the fish first? Justify your answer.

68. USING TOOLS You are asked to find a possible pair of integer values for a and c so that the equation $ax^2 - 3x + c = 0$ has two real solutions. When you solve the inequality for the discriminant, you obtain $ac < 2.25$. So, you choose the values $a = 2$ and $c = 1$. Your graphing calculator displays the graph of your equation in a standard viewing window. Is your solution correct? Explain.

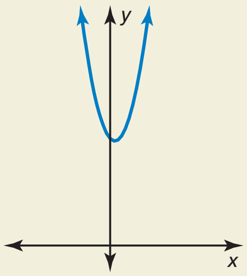


69. PROBLEM SOLVING Your family has a rectangular pool that measures 18 feet by 9 feet. Your family wants to put a deck around the pool but is not sure how wide to make the deck. Determine how wide the deck should be when the total area of the pool and deck is 400 square feet. What is the width of the deck?

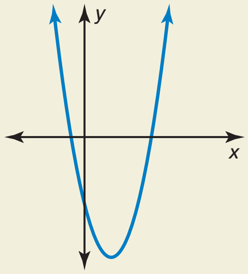


70. HOW DO YOU SEE IT? The graph of a quadratic function $y = ax^2 + bx + c$ is shown. Determine whether each discriminant of $ax^2 + bx + c = 0$ is *positive*, *negative*, or *zero*. Then state the number and type of solutions for each graph. Explain your reasoning.

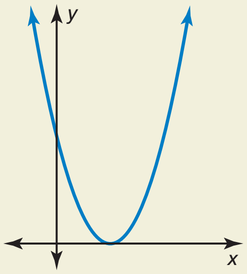
a.



b.



c.



71. CRITICAL THINKING Solve each absolute value equation.

a. $|x^2 - 3x - 14| = 4$ **b.** $x^2 = |x| + 6$

72. MAKING AN ARGUMENT The class is asked to solve the equation $4x^2 + 14x + 11 = 0$. You decide to solve the equation by completing the square. Your friend decides to use the Quadratic Formula. Whose method is more efficient? Explain your reasoning.

73. ABSTRACT REASONING For a quadratic equation $ax^2 + bx + c = 0$ with two real solutions, show that the mean of the solutions is $-\dfrac{b}{2a}$. How is this fact related to the symmetry of the graph of $y = ax^2 + bx + c$?

74. THOUGHT PROVOKING Describe a real-life story that could be modeled by $h = -16t^2 + v_0t + h_0$. Write the height model for your story and determine how long your object is in the air.

75. REASONING Show there is no quadratic equation $ax^2 + bx + c = 0$ such that a, b, and c are real numbers and $3i$ and $-2i$ are solutions.

76. MODELING WITH MATHEMATICS The Stratosphere Tower in Las Vegas is 921 feet tall and has a "needle" at its top that extends even higher into the air. A thrill ride called Big Shot catapults riders 160 feet up the needle and then lets them fall back to the launching pad.

a. The height h (in feet) of a rider on the Big Shot can be modeled by $h = -16t^2 + v_0t + 921$, where t is the elapsed time (in seconds) after launch and v_0 is the initial vertical velocity (in feet per second). Find v_0 using the fact that the maximum value of h is $921 + 160 = 1081$ feet.

b. A brochure for the Big Shot states that the ride up the needle takes 2 seconds. Compare this time to the time given by the model $h = -16t^2 + v_0t + 921$, where v_0 is the value you found in part (a). Discuss the accuracy of the model.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations by graphing. *(Skills Review Handbook)*

77. $-x + 2y = 6$
$x + 4y = 24$

78. $y = 2x - 1$
$y = x + 1$

79. $3x + y = 4$
$6x + 2y = -4$

80. $y = -x + 2$
$-5x + 5y = 10$

Graph the quadratic equation. Label the vertex and axis of symmetry. *(Section 2.2)*

81. $y = -x^2 + 2x + 1$

82. $y = 2x^2 - x + 3$

83. $y = 0.5x^2 + 2x + 5$

84. $y = -3x^2 - 2$