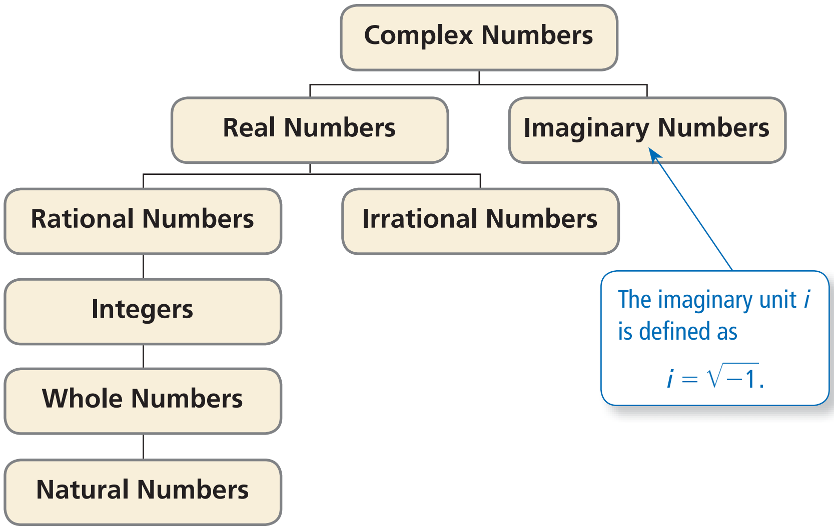
3.2 Complex Numbers

Essential Question

What are the subsets of the set of complex numbers?

In your study of mathematics, you have probably worked with only *real numbers*, which can be represented graphically on the real number line. In this lesson, the system of numbers is expanded to include *imaginary numbers*. The real numbers and imaginary numbers compose the set of *complex numbers*.



EXPLORATION 1 Classifying Numbers

Work with a partner. Determine which subsets of the set of complex numbers contain each number.

a. $\sqrt{9}$ **b.** $\sqrt{0}$ **c.** $-\sqrt{4}$

d. $\sqrt{\frac{4}{9}}$ **e.** $\sqrt{2}$ **f.** $\sqrt{-1}$

ATTENDING TO PRECISION

To be proficient in math, you need to use clear definitions in your reasoning and discussions with others.

EXPLORATION 2 Complex Solutions of Quadratic Equations

Work with a partner. Use the definition of the imaginary unit i to match each quadratic equation with its complex solution. Justify your answers.

a. $x^2 - 4 = 0$ **b.** $x^2 + 1 = 0$ **c.** $x^2 - 1 = 0$

d. $x^2 + 4 = 0$ **e.** $x^2 - 9 = 0$ **f.** $x^2 + 9 = 0$

A. i **B.** $3i$ **C.** 3

D. $2i$ **E.** 1 **F.** 2

Communicate Your Answer

3. What are the subsets of the set of complex numbers? Give an example of a number in each subset.

4. Is it possible for a number to be both whole and natural? natural and rational? rational and irrational? real and imaginary? Explain your reasoning.

3.2 Lesson

Core Vocabulary

imaginary unit *i*, *p. 104*
complex number, *p. 104*
imaginary number, *p. 104*
pure imaginary number, *p. 104*

What You Will Learn

- Define and use the imaginary unit *i*.
- Add, subtract, and multiply complex numbers.
- Find complex solutions and zeros.

The Imaginary Unit *i*

Not all quadratic equations have real-number solutions. For example, $x^2 = -3$ has no real-number solutions because the square of any real number is never a negative number.

To overcome this problem, mathematicians created an expanded system of numbers using the **imaginary unit *i***, defined as $i = \sqrt{-1}$. Note that $i^2 = -1$. The imaginary unit i can be used to write the square root of *any* negative number.

Core Concept

The Square Root of a Negative Number

Property	Example
1. If r is a positive real number, then $\sqrt{-r} = i\sqrt{r}$.	$\sqrt{-3} = i\sqrt{3}$
2. By the first property, it follows that $(i\sqrt{r})^2 = -r$.	$(i\sqrt{3})^2 = i^2 \cdot 3 = -3$

EXAMPLE 1 Finding Square Roots of Negative Numbers

Find the square root of each number.

a. $\sqrt{-25}$ **b.** $\sqrt{-72}$ **c.** $-5\sqrt{-9}$

SOLUTION

a. $\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} = 5i$

b. $\sqrt{-72} = \sqrt{72} \cdot \sqrt{-1} = \sqrt{36} \cdot \sqrt{2} \cdot i = 6\sqrt{2}\, i = 6i\sqrt{2}$

c. $-5\sqrt{-9} = -5\sqrt{9} \cdot \sqrt{-1} = -5 \cdot 3 \cdot i = -15i$

Monitoring Progress

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Find the square root of the number.

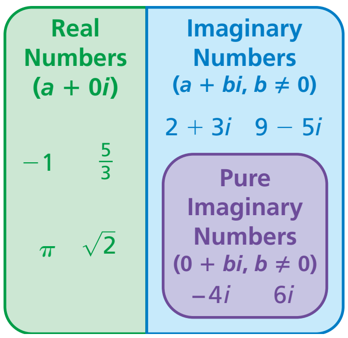
1. $\sqrt{-4}$ **2.** $\sqrt{-12}$ **3.** $-\sqrt{-36}$ **4.** $2\sqrt{-54}$

A **complex number** written in *standard form* is a number $a + bi$ where a and b are real numbers. The number a is the *real part*, and the number bi is the *imaginary part*.

$$a + bi$$

If $b \neq 0$, then $a + bi$ is an **imaginary number**. If $a = 0$ and $b \neq 0$, then $a + bi$ is a **pure imaginary number**. The diagram shows how different types of complex numbers are related.

Complex Numbers ($a + bi$)



Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

EXAMPLE 2 Equality of Two Complex Numbers

Find the values of x and y that satisfy the equation $2x - 7i = 10 + yi$.

SOLUTION

Set the real parts equal to each other and the imaginary parts equal to each other.

$2x = 10$	Equate the real parts.	$-7i = yi$	Equate the imaginary parts.
$x = 5$	Solve for x.	$-7 = y$	Solve for y.

▶ So, $x = 5$ and $y = -7$.

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Find the values of x and y that satisfy the equation.

5. $x + 3i = 9 - yi$

6. $9 + 4yi = -2x + 3i$

Operations with Complex Numbers

Core Concept

Sums and Differences of Complex Numbers

To add (or subtract) two complex numbers, add (or subtract) their real parts and their imaginary parts separately.

Sum of complex numbers: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference of complex numbers: $(a + bi) - (c + di) = (a - c) + (b - d)i$

EXAMPLE 3 Adding and Subtracting Complex Numbers

Add or subtract. Write the answer in standard form.

a. $(8 - i) + (5 + 4i)$

b. $(7 - 6i) - (3 - 6i)$

c. $13 - (2 + 7i) + 5i$

SOLUTION

a. $(8 - i) + (5 + 4i) = (8 + 5) + (-1 + 4)i$ — Definition of complex addition

$= 13 + 3i$ — Write in standard form.

b. $(7 - 6i) - (3 - 6i) = (7 - 3) + (-6 + 6)i$ — Definition of complex subtraction

$= 4 + 0i$ — Simplify.

$= 4$ — Write in standard form.

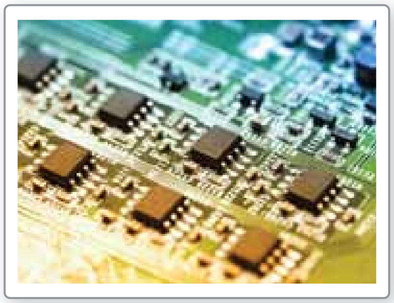
c. $13 - (2 + 7i) + 5i = [(13 - 2) - 7i] + 5i$ — Definition of complex subtraction

$= (11 - 7i) + 5i$ — Simplify.

$= 11 + (-7 + 5)i$ — Definition of complex addition

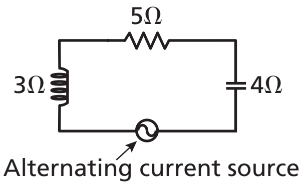
$= 11 - 2i$ — Write in standard form.

EXAMPLE 4 Solving a Real-Life Problem



Electrical circuit components, such as resistors, inductors, and capacitors, all oppose the flow of current. This opposition is called *resistance* for resistors and *reactance* for inductors and capacitors. Each of these quantities is measured in ohms. The symbol used for ohms is Ω, the uppercase Greek letter omega.

Component and symbol	Resistor	Inductor	Capacitor
Resistance or reactance (in ohms)	R	L	C
Impedance (in ohms)	R	Li	$-Ci$



The table shows the relationship between a component's resistance or reactance and its contribution to impedance. A *series circuit* is also shown with the resistance or reactance of each component labeled. The impedance for a series circuit is the sum of the impedances for the individual components. Find the impedance of the circuit.

SOLUTION

The resistor has a resistance of 5 ohms, so its impedance is 5 ohms. The inductor has a reactance of 3 ohms, so its impedance is $3i$ ohms. The capacitor has a reactance of 4 ohms, so its impedance is $-4i$ ohms.

$$\text{Impedance of circuit} = 5 + 3i + (-4i) = 5 - i$$

▶ The impedance of the circuit is $(5 - i)$ ohms.

To multiply two complex numbers, use the Distributive Property, or the FOIL method, just as you do when multiplying real numbers or algebraic expressions.

EXAMPLE 5 Multiplying Complex Numbers

STUDY TIP

When simplifying an expression that involves complex numbers, be sure to simplify i^2 as -1.

Multiply. Write the answer in standard form.

a. $4i(-6 + i)$ **b.** $(9 - 2i)(-4 + 7i)$

SOLUTION

a. $4i(-6 + i) = -24i + 4i^2$ Distributive Property

$= -24i + 4(-1)$ Use $i^2 = -1$.

$= -4 - 24i$ Write in standard form.

b. $(9 - 2i)(-4 + 7i) = -36 + 63i + 8i - 14i^2$ Multiply using FOIL.

$= -36 + 71i - 14(-1)$ Simplify and use $i^2 = -1$.

$= -36 + 71i + 14$ Simplify.

$= -22 + 71i$ Write in standard form.

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7. **WHAT IF?** In Example 4, what is the impedance of the circuit when the capacitor is replaced with one having a reactance of 7 ohms?

Perform the operation. Write the answer in standard form.

8. $(9 - i) + (-6 + 7i)$
9. $(3 + 7i) - (8 - 2i)$
10. $-4 - (1 + i) - (5 + 9i)$
11. $(-3i)(10i)$
12. $i(8 - i)$
13. $(3 + i)(5 - i)$

Complex Solutions and Zeros

EXAMPLE 6 Solving Quadratic Equations

Solve (a) $x^2 + 4 = 0$ and (b) $2x^2 - 11 = -47$.

SOLUTION

LOOKING FOR STRUCTURE

Notice that you can use the solutions in Example 6(a) to factor $x^2 + 4$ as $(x + 2i)(x - 2i)$.

a. $x^2 + 4 = 0$ — Write original equation.

$x^2 = -4$ — Subtract 4 from each side.

$x = \pm\sqrt{-4}$ — Take square root of each side.

$x = \pm 2i$ — Write in terms of i.

▶ The solutions are $2i$ and $-2i$.

b. $2x^2 - 11 = -47$ — Write original equation.

$2x^2 = -36$ — Add 11 to each side.

$x^2 = -18$ — Divide each side by 2.

$x = \pm\sqrt{-18}$ — Take square root of each side.

$x = \pm i\sqrt{18}$ — Write in terms of i.

$x = \pm 3i\sqrt{2}$ — Simplify radical.

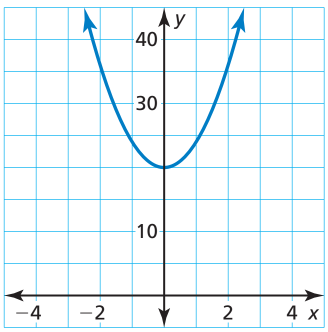
▶ The solutions are $3i\sqrt{2}$ and $-3i\sqrt{2}$.

EXAMPLE 7 Finding Zeros of a Quadratic Function

Find the zeros of $f(x) = 4x^2 + 20$.

FINDING AN ENTRY POINT

The graph of f does not intersect the x-axis, which means f has no real zeros. So, f must have complex zeros, which you can find algebraically.



SOLUTION

$4x^2 + 20 = 0$ — Set $f(x)$ equal to 0.

$4x^2 = -20$ — Subtract 20 from each side.

$x^2 = -5$ — Divide each side by 4.

$x = \pm\sqrt{-5}$ — Take square root of each side.

$x = \pm i\sqrt{5}$ — Write in terms of i.

▶ So, the zeros of f are $i\sqrt{5}$ and $-i\sqrt{5}$.

Check

$f(i\sqrt{5}) = 4(i\sqrt{5})^2 + 20 = 4 \cdot 5i^2 + 20 = 4(-5) + 20 = 0$ ✓

$f(-i\sqrt{5}) = 4(-i\sqrt{5})^2 + 20 = 4 \cdot 5i^2 + 20 = 4(-5) + 20 = 0$ ✓

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Solve the equation.

14. $x^2 = -13$

15. $x^2 = -38$

16. $x^2 + 11 = 3$

17. $x^2 - 8 = -36$

18. $3x^2 - 7 = -31$

19. $5x^2 + 33 = 3$

Find the zeros of the function.

20. $f(x) = x^2 + 7$

21. $f(x) = -x^2 - 4$

22. $f(x) = 9x^2 + 1$

3.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is the imaginary unit i defined as and how can you use i?

2. **COMPLETE THE SENTENCE** For the complex number $5 + 2i$, the imaginary part is ____ and the real part is ____.

3. **WRITING** Describe how to add complex numbers.

4. **WHICH ONE DOESN'T BELONG?** Which number does *not* belong with the other three? Explain your reasoning.

$3 + 0i$ $\quad$ $2 + 5i$ $\quad$ $\sqrt{3} + 6i$ $\quad$ $0 - 7i$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, find the square root of the number. *(See Example 1.)*

5. $\sqrt{-36}$
6. $\sqrt{-64}$
7. $\sqrt{-18}$
8. $\sqrt{-24}$
9. $2\sqrt{-16}$
10. $-3\sqrt{-49}$
11. $-4\sqrt{-32}$
12. $6\sqrt{-63}$

In Exercises 13–20, find the values of x and y that satisfy the equation. *(See Example 2.)*

13. $4x + 2i = 8 + yi$
14. $3x + 6i = 27 + yi$
15. $-10x + 12i = 20 + 3yi$
16. $9x - 18i = -36 + 6yi$
17. $2x - yi = 14 + 12i$
18. $-12x + yi = 60 - 13i$
19. $54 - \frac{1}{7}yi = 9x - 4i$
20. $15 - 3yi = \frac{1}{2}x + 2i$

In Exercises 21–30, add or subtract. Write the answer in standard form. *(See Example 3.)*

21. $(6 - i) + (7 + 3i)$
22. $(9 + 5i) + (11 + 2i)$
23. $(12 + 4i) - (3 - 7i)$
24. $(2 - 15i) - (4 + 5i)$
25. $(12 - 3i) + (7 + 3i)$
26. $(16 - 9i) - (2 - 9i)$
27. $7 - (3 + 4i) + 6i$
28. $16 - (2 - 3i) - i$
29. $-10 + (6 - 5i) - 9i$
30. $-3 + (8 + 2i) + 7i$

31. **USING STRUCTURE** Write each expression as a complex number in standard form.

    a. $\sqrt{-9} + \sqrt{-4} - \sqrt{16}$

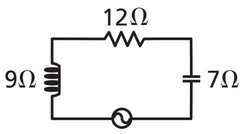
    b. $\sqrt{-16} + \sqrt{8} + \sqrt{-36}$

32. **REASONING** The additive inverse of a complex number z is a complex number z_a such that $z + z_a = 0$. Find the additive inverse of each complex number.

    a. $z = 1 + i$ $\quad$ **b.** $z = 3 - i$ $\quad$ **c.** $z = -2 + 8i$

In Exercises 33–36, find the impedance of the series circuit. *(See Example 4.)*

33.



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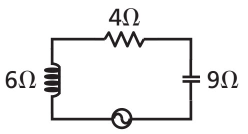

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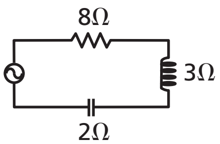

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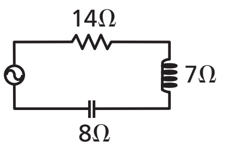


In Exercises 37–44, multiply. Write the answer in standard form. *(See Example 5.)*

37. $3i(-5 + i)$

38. $2i(7 - i)$

39. $(3 - 2i)(4 + i)$

40. $(7 + 5i)(8 - 6i)$

41. $(4 - 2i)(4 + 2i)$

42. $(9 + 5i)(9 - 5i)$

43. $(3 - 6i)^2$

44. $(8 + 3i)^2$

JUSTIFYING STEPS **In Exercises 45 and 46, justify each step in performing the operation.**

45. $11 - (4 + 3i) + 5i$

$= [(11 - 4) - 3i] + 5i$

$= (7 - 3i) + 5i$

$= 7 + (-3 + 5)i$

$= 7 + 2i$

46. $(3 + 2i)(7 - 4i)$

$= 21 - 12i + 14i - 8i^2$

$= 21 + 2i - 8(-1)$

$= 21 + 2i + 8$

$= 29 + 2i$

REASONING **In Exercises 47 and 48, place the tiles in the expression to make a true statement.**

47. $(\_\_\_\_ - \_\_\_\_i) - (\_\_\_\_ - \_\_\_\_i) = 2 - 4i$

7 4 3 6

48. $\_\_\_\_i(\_\_\_\_ + \_\_\_\_i) = -18 - 10i$

−5 9 2

In Exercises 49–54, solve the equation. Check your solution(s). *(See Example 6.)*

49. $x^2 + 9 = 0$

50. $x^2 + 49 = 0$

51. $x^2 - 4 = -11$

52. $x^2 - 9 = -15$

53. $2x^2 + 6 = -34$

54. $x^2 + 7 = -47$

In Exercises 55–62, find the zeros of the function. *(See Example 7.)*

55. $f(x) = 3x^2 + 6$

56. $g(x) = 7x^2 + 21$

57. $h(x) = 2x^2 + 72$

58. $k(x) = -5x^2 - 125$

59. $m(x) = -x^2 - 27$

60. $p(x) = x^2 + 98$

61. $r(x) = -\frac{1}{2}x^2 - 24$

62. $f(x) = -\frac{1}{5}x^2 - 10$

ERROR ANALYSIS **In Exercises 63 and 64, describe and correct the error in performing the operation and writing the answer in standard form.**

63.

$$\begin{aligned}(3 + 2i)(5 - i) &= 15 - 3i + 10i - 2i^2 \\ &= 15 + 7i - 2i^2 \\ &= -2i^2 + 7i + 15\end{aligned}$$

64.

$$\begin{aligned}(4 + 6i)^2 &= (4)^2 + (6i)^2 \\ &= 16 + 36i^2 \\ &= 16 + (36)(-1) \\ &= -20\end{aligned}$$

65. **NUMBER SENSE** Simplify each expression. Then classify your results in the table below.

a. $(-4 + 7i) + (-4 - 7i)$

b. $(2 - 6i) - (-10 + 4i)$

c. $(25 + 15i) - (25 - 6i)$

d. $(5 + i)(8 - i)$

e. $(17 - 3i) + (-17 - 6i)$

f. $(-1 + 2i)(11 - i)$

g. $(7 + 5i) + (7 - 5i)$

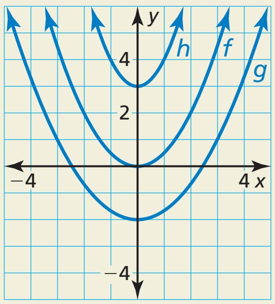
h. $(-3 + 6i) - (-3 - 8i)$

Real numbers	Imaginary numbers	Pure imaginary numbers

66. **MAKING AN ARGUMENT** The Product Property of Square Roots states $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. Your friend concludes $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{36} = 6$. Is your friend correct? Explain.

67. **FINDING A PATTERN** Make a table that shows the powers of i from i^1 to i^8 in the first row and the simplified forms of these powers in the second row. Describe the pattern you observe in the table. Verify the pattern continues by evaluating the next four powers of i.

68. **HOW DO YOU SEE IT?** The graphs of three functions are shown. Which function(s) has real zeros? imaginary zeros? Explain your reasoning.



In Exercises 69–74, write the expression as a complex number in standard form.

69. $(3 + 4i) - (7 - 5i) + 2i(9 + 12i)$

70. $3i(2 + 5i) + (6 - 7i) - (9 + i)$

71. $(3 + 5i)(2 - 7i^4)$

72. $2i^3(5 - 12i)$

73. $(2 + 4i^5) + (1 - 9i^6) - (3 + i^7)$

74. $(8 - 2i^4) + (3 - 7i^8) - (4 + i^9)$

75. **OPEN-ENDED** Find two imaginary numbers whose sum and product are real numbers. How are the imaginary numbers related?

76. **COMPARING METHODS** Describe the two different methods shown for writing the complex expression in standard form. Which method do you prefer? Explain.

Method 1

$$\begin{aligned} 4i(2 - 3i) + 4i(1 - 2i) &= 8i - 12i^2 + 4i - 8i^2 \\ &= 8i - 12(-1) + 4i - 8(-1) \\ &= 20 + 12i \end{aligned}$$

Method 2

$$\begin{aligned} 4i(2 - 3i) + 4i(1 - 2i) &= 4i[(2 - 3i) + (1 - 2i)] \\ &= 4i[3 - 5i] \\ &= 12i - 20i^2 \\ &= 12i - 20(-1) \\ &= 20 + 12i \end{aligned}$$

77. **CRITICAL THINKING** **Determine whether each statement is *true* or *false*. If it is true, give an example. If it is false, give a counterexample.**

a. The sum of two imaginary numbers is an imaginary number.

b. The product of two pure imaginary numbers is a real number.

c. A pure imaginary number is an imaginary number.

d. A complex number is a real number.

78. **THOUGHT PROVOKING** Create a circuit that has an impedance of $14 - 3i$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the given value of x is a solution to the equation. *(Skills Review Handbook)*

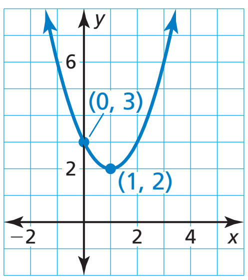
79. $3(x - 2) + 4x - 1 = x - 1;\ x = 1$

80. $x^3 - 6 = 2x^2 + 9 - 3x;\ x = -5$

81. $-x^2 + 4x = \frac{19}{3}x^2;\ x = -\frac{3}{4}$

Write a quadratic function in vertex form whose graph is shown. *(Section 2.4)*

82.



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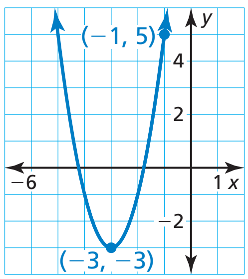

84.