3.2 Complex Numbers

Essential Question What are the subsets of the set of complex numbers?

In your study of mathematics, you have probably worked with only real numbers, which can be represented graphically on the real number line. In this lesson, the system of numbers is expanded to include imaginary numbers. The real numbers and imaginary numbers compose the set of complex numbers.

Complex Numbers

Real Numbers

Imaginary Numbers

Rational Numbers

Irrational Numbers

Integers

Whole Numbers

Natural Numbers

EXPLORATION 1 Classifying Numbers

Work with a partner. Determine which subsets of the set of complex numbers contain each number.

a. $\sqrt{9}$

b. $\sqrt{0}$

c. $-\sqrt{4}$

d. $\frac{4}{\sqrt{9}}$

e. $\sqrt{2}$

f. $\sqrt{-1}$

EXPLORATION 2 Complex Solutions of Quadratic Equations

Work with a partner. Use the definition of the imaginary unit $i$ to match each quadratic equation with its complex solution. Justify your answers.

a. $x^2 - 4 = 0$

b. $x^2 + 1 = 0$

c. $x^2 - 1 = 0$

d. $x^2 + 4 = 0$

e. $x^2 - 9 = 0$

f. $x^2 + 9 = 0$

A. $i$

B. $3i$

C. 3

D. $2i$

E. 1

F. 2

Communicate Your Answer

3. What are the subsets of the set of complex numbers? Give an example of a number in each subset.

4. Is it possible for a number to be both whole and natural? natural and rational? rational and irrational? real and imaginary? Explain your reasoning.
What You Will Learn

- Define and use the imaginary unit \( i \).
- Add, subtract, and multiply complex numbers.
- Find complex solutions and zeros.

## The Imaginary Unit \( i \)

Not all quadratic equations have real-number solutions. For example, \( x^2 = -3 \) has no real-number solutions because the square of any real number is never a negative number.

To overcome this problem, mathematicians created an expanded system of numbers using the imaginary unit \( i \), defined as \( i = \sqrt{-1} \). Note that \( i^2 = -1 \). The imaginary unit \( i \) can be used to write the square root of any negative number.

### Core Concept

**The Square Root of a Negative Number**

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#### Example 1: Finding Square Roots of Negative Numbers

Find the square root of each number.

- **a.** \( \sqrt{-25} \)
  - **SOLUTION**
    \( \sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} = 5i \)

- **b.** \( \sqrt{-72} \)
  - **SOLUTION**
    \( \sqrt{-72} = \sqrt{72} \cdot \sqrt{-1} = \sqrt{36 \cdot 2} \cdot i = 6\sqrt{2} i = 6i\sqrt{2} \)

- **c.** \( -5\sqrt{-9} \)
  - **SOLUTION**
    \( -5\sqrt{-9} = -5\sqrt{9} \cdot \sqrt{-1} = -5 \cdot 3 \cdot i = -15i \)

### Monitoring Progress

Find the square root of the number.

1. \( \sqrt{-4} \)
2. \( \sqrt{-12} \)
3. \( -\sqrt{-36} \)
4. \( 2\sqrt{-54} \)

A complex number written in standard form is a number \( a + bi \) where \( a \) and \( b \) are real numbers. The number \( a \) is the real part, and the number \( bi \) is the imaginary part.

- **Complex Numbers**

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- **Pure Imaginary Numbers**
  - \( 0 + bi, b \neq 0 \)

### Core Vocabulary

- Imaginary unit \( i \), p. 104
- Complex number, p. 104
- Imaginary number, p. 104
- Pure imaginary number, p. 104

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**What You Will Learn**

- Define and use the imaginary unit \( i \).
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  - **SOLUTION**
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- **c.** \( -5\sqrt{-9} \)
  - **SOLUTION**
    \( -5\sqrt{-9} = -5\sqrt{9} \cdot \sqrt{-1} = -5 \cdot 3 \cdot i = -15i \)

### Monitoring Progress

Find the square root of the number.

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2. \( \sqrt{-12} \)
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- **Pure Imaginary Numbers**
  - \( 0 + bi, b \neq 0 \)
Adding and Subtracting Complex Numbers

a. \((8 - i) + (5 + 4i)\)

\[= (8 + 5) + (-1 + 4)i\]
\[= 13 + 3i\]

b. \((7 - 6i) - (3 - 6i)\)

\[= (7 - 3) + (-6 + 6)i\]
\[= 4 + 0i\]
\[= 4\]

c. \(13 - (2 + 7i) + 5i\)

\[= [(13 - 2) - 7i] + 5i\]
\[= (11 - 7i) + 5i\]
\[= 11 + (-7 + 5)i\]
\[= 11 - 2i\]
### Example 4 Solving a Real-Life Problem

Electrical circuit components, such as resistors, inductors, and capacitors, all oppose the flow of current. This opposition is called **resistance** for resistors and **reactance** for inductors and capacitors. Each of these quantities is measured in ohms. The symbol used for ohms is Ω, the uppercase Greek letter omega.

<table>
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<th>Component and symbol</th>
<th>Resistor</th>
<th>Inductor</th>
<th>Capacitor</th>
</tr>
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<tr>
<td>Resistance or reactance (in ohms)</td>
<td>R</td>
<td>L</td>
<td>C</td>
</tr>
<tr>
<td>Impedance (in ohms)</td>
<td>R</td>
<td>Li</td>
<td>−Ci</td>
</tr>
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![Electrical circuit components](image)

The table shows the relationship between a component’s resistance or reactance and its contribution to impedance. A **series circuit** is also shown with the resistance or reactance of each component labeled. The impedance for a series circuit is the sum of the impedances for the individual components. Find the impedance of the circuit.

**SOLUTION**

The resistor has a resistance of 5 ohms, so its impedance is 5 ohms. The inductor has a reactance of 3 ohms, so its impedance is 3i ohms. The capacitor has a reactance of 4 ohms, so its impedance is −4i ohms.

\[
\text{Impedance of circuit} = 5 + 3i + (−4i) = 5 − i
\]

The impedance of the circuit is (5 − i) ohms.

To multiply two complex numbers, use the Distributive Property, or the FOIL method, just as you do when multiplying real numbers or algebraic expressions.

### Example 5 Multiplying Complex Numbers

Multiply. Write the answer in standard form.

a. \(4i(−6 + i)\)  
b. \((9 − 2i)(−4 + 7i)\)

**SOLUTION**

a. \(4i(−6 + i) = −24i + 4i^2\)
   
   \(= −24i + 4(−1)\)
   
   \(= −4 − 24i\)

b. \((9 − 2i)(−4 + 7i) = −36 + 63i + 8i − 14i^2\)
   
   \(= −36 + 71i − 14(−1)\)
   
   \(= −36 + 71i + 14\)
   
   \(= −22 + 71i\)

### Monitoring Progress

7. **WHAT IF?** In Example 4, what is the impedance of the circuit when the capacitor is replaced with one having a reactance of 7 ohms?

Perform the operation. Write the answer in standard form.

8. \((9 − i)(−6 + 7i)\)  
9. \((3 + 7i) − (8 − 2i)\)  
10. \(−4 − (1 + i) − (5 + 9i)\)  
11. \((−3i)(10i)\)  
12. \(i(8 − i)\)  
13. \((3 + i)(5 − i)\)
Complex Solutions and Zeros

**Example 6** Solving Quadratic Equations

Solve (a) \(x^2 + 4 = 0\) and (b) \(2x^2 - 11 = -47\).

**Solution**

\[a. \quad x^2 + 4 = 0\]

Write original equation. Subtract 4 from each side.

\[x^2 = -4\]

Take square root of each side. Write in terms of \(i\).

\[x = \pm \sqrt{-4} = \pm 2i\]

The solutions are \(2i\) and \(-2i\).

\[b. \quad 2x^2 - 11 = -47\]

Write original equation. Add 11 to each side. Divide each side by 2.

\[2x^2 = -36\]

\[x^2 = -18\]

Take square root of each side. Write in terms of \(i\). Simplify radical.

\[x = \pm \sqrt{-18} = \pm i\sqrt{18} = \pm 3i\sqrt{2}\]

The solutions are \(3i\sqrt{2}\) and \(-3i\sqrt{2}\).

**Example 7** Finding Zeros of a Quadratic Function

Find the zeros of \(f(x) = 4x^2 + 20\).

**Solution**

\[4x^2 + 20 = 0\]

Set \(f(x)\) equal to 0. Subtract 20 from each side. Divide each side by 4. Take square root of each side. Write in terms of \(i\).

\[4x^2 = -20\]

\[x^2 = -5\]

\[x = \pm \sqrt{-5} = \pm i\sqrt{5}\]

So, the zeros of \(f\) are \(i\sqrt{5}\) and \(-i\sqrt{5}\).

**Check**

\[f(i\sqrt{5}) = 4(i\sqrt{5})^2 + 20 = 4 \cdot 5i^2 + 20 = 4(-5) + 20 = 0 \checkmark\]

\[f(-i\sqrt{5}) = 4(-i\sqrt{5})^2 + 20 = 4 \cdot 5i^2 + 20 = 4(-5) + 20 = 0 \checkmark\]

**Monitoring Progress**

Help in English and Spanish at BigIdeasMath.com

Solve the equation.

14. \(x^2 = -13\) 15. \(x^2 = -38\) 16. \(x^2 + 11 = 3\)
17. \(x^2 - 8 = -36\) 18. \(3x^2 - 7 = -31\) 19. \(5x^2 + 33 = 3\)

Find the zeros of the function.

20. \(f(x) = x^2 + 7\) 21. \(f(x) = -x^2 - 4\) 22. \(f(x) = 9x^2 + 1\)

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3.2 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** What is the imaginary unit $i$ defined as and how can you use $i$?

2. **COMPLETE THE SENTENCE** For the complex number $5 + 2i$, the imaginary part is ____ and the real part is ____.

3. **WRITING** Describe how to add complex numbers.

4. **WHICH ONE DOESN'T BELONG?** Which number does not belong with the other three? Explain your reasoning.

3 + 0i  2 + 5i  √3 + 6i  0 − 7i

**Monitoring Progress and Modeling with Mathematics**

In Exercises 5–12, find the square root of the number. (See Example 1.)

5. $\sqrt{-36}$  6. $\sqrt{-64}$

7. $\sqrt{-18}$  8. $\sqrt{-24}$

9. $2\sqrt{-16}$  10. $-3\sqrt{-49}$

11. $-4\sqrt{32}$  12. $6\sqrt{-63}$

In Exercises 13–20, find the values of $x$ and $y$ that satisfy the equation. (See Example 2.)

13. $4x + 2i = 8 + yi$

14. $3x + 6i = 27 + yi$

15. $-10x + 12i = 20 + 3yi$

16. $9x - 18i = -36 + 6yi$

17. $2x - yi = 14 + 12i$

18. $-12x + yi = 60 - 13i$

19. $54 - \frac{3}{2}yi = 9x - 4i$

20. $15 - 3yi = \frac{1}{3}x + 2i$

In Exercises 21–30, add or subtract. Write the answer in standard form. (See Example 3.)

21. $(6 - i) + (7 + 3i)$

22. $(9 + 5i) + (11 + 2i)$

23. $(12 + 4i) - (3 - 7i)$

24. $(2 - 15i) - (4 + 5i)$

25. $(12 - 3i) + (7 + 3i)$

26. $(16 - 9i) - (2 - 9i)$

27. $7 - (3 + 4i) + 6i$

28. $16 - (2 - 3i) - i$

29. $-10 + (6 - 5i) - 9i$

30. $-3 + (8 + 2i) + 7i$

31. **USING STRUCTURE** Write each expression as a complex number in standard form.

   a. $\sqrt{-9} + \sqrt{-4} - \sqrt{16}$
   b. $\sqrt{-16} + \sqrt{8} + \sqrt{-36}$

32. **REASONING** The additive inverse of a complex number $z$ is a complex number $z_a$ such that $z + z_a = 0$. Find the additive inverse of each complex number.

   a. $z = 1 + i$
   b. $z = 3 - i$
   c. $z = -2 + 8i$

In Exercises 33–36, find the impedance of the series circuit. (See Example 4.)

33. [Circuit Diagram]

34. [Circuit Diagram]

35. [Circuit Diagram]

36. [Circuit Diagram]
In Exercises 37–44, multiply. Write the answer in standard form. (See Example 5.)
37. \(3i(5 + i)\)  
38. \(2(7 - i)\)
39. \((3 - 2i)(4 + i)\)  
40. \((7 + 5i)(8 - 6i)\)
41. \((4 - 2i)(4 + 2i)\)  
42. \((9 + 5i)(9 - 5i)\)
43. \((3 - 6i)^2\)  
44. \((8 + 3i)^2\)

**JUSTIFYING STEPS** In Exercises 45 and 46, justify each step in performing the operation.

45. \(11 - (4 + 3i) + 5i\)
   \[\begin{align*}
   &= [(11 - 4) - 3i] + 5i \\
   &= (7 - 3i) + 5i \\
   &= 7 + (-3 + 5)i \\
   &= 7 + 2i \\
\end{align*}\]

46. \((3 + 2i)(7 - 4i)\)
   \[\begin{align*}
   &= 21 - 12i + 14i - 8i^2 \\
   &= 21 + 2i - 8(-1) \\
   &= 21 + 2i + 8 \\
   &= 29 + 2i \\
\end{align*}\]

**REASONING** In Exercises 47 and 48, place the tiles in the expression to make a true statement.

47. \(_{\text{____}} - (\text{____} - \text{____}i) = 2 - 4i\)
   \[\begin{align*}
   7 & \quad 4 & \quad 3 & \quad 6 \\
\end{align*}\]

48. \(\text{____}i(\text{____} + \text{____}i) = -18 - 10i\)
   \[\begin{align*}
   -5 & \quad 9 & \quad 2 \\
\end{align*}\]

In Exercises 49–54, solve the equation. Check your solution(s). (See Example 6.)
49. \(x^2 + 9 = 0\)  
50. \(x^2 + 49 = 0\)
51. \(x^2 - 4 = -11\)
52. \(x^2 - 9 = -15\)
53. \(2x^2 + 6 = -34\)
54. \(x^2 + 7 = -47\)

In Exercises 55–62, find the zeros of the function. (See Example 7.)
55. \(f(x) = 3x^2 + 6\)  
56. \(g(x) = 7x^2 + 21\)
57. \(h(x) = 2x^2 + 72\)  
58. \(k(x) = -5x^2 - 125\)
59. \(m(x) = -x^2 - 27\)  
60. \(p(x) = x^2 + 98\)
61. \(r(x) = -\frac{1}{2}x^2 - 24\)  
62. \(f(x) = -\frac{1}{2}x^2 - 10\)

**ERROR ANALYSIS** In Exercises 63 and 64, describe and correct the error in performing the operation and writing the answer in standard form.

63. \(3 + 2i)(5 - i) = 15 - 3i + 10i - 2i^2\)
   \[\begin{align*}
   &= 15 + 7i - 2i^2 \\
   &= -2i^2 + 7i + 15 \\
\end{align*}\]

64. \((4 + 6i)^2 = (4)^2 + (6i)^2\)
   \[\begin{align*}
   &= 16 + 36i^2 \\
   &= 16 + (36)(-1) \\
   &= -20 \\
\end{align*}\]

**NUMBER SENSE** Simplify each expression. Then classify your results in the table below.

a. \((-4 + 7i) + (-4 - 7i)\)
b. \((2 - 6i) - (-10 + 4i)\)
c. \((25 + 15i) - (25 - 6i)\)
d. \((5 + i)(8 - i)\)
e. \((17 - 3i) + (-17 - 6i)\)
f. \((-1 + 2i)(11 - i)\)
g. \((7 + 5i) + (7 - 5i)\)
h. \((-3 + 6i) - (-3 - 8i)\)

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**MAKING AN ARGUMENT** The Product Property of Square Roots states \(\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}\). Your friend concludes \(\sqrt{-4} \cdot \sqrt{-9} = \sqrt{36} = 6\). Is your friend correct? Explain.

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67. **FINDING A PATTERN** Make a table that shows the powers of \( i \) from \( i^1 \) to \( i^8 \) in the first row and the simplified forms of these powers in the second row. Describe the pattern you observe in the table. Verify the pattern continues by evaluating the next four powers of \( i \).

68. **HOW DO YOU SEE IT?** The graphs of three functions are shown. Which function(s) has real zeros? imaginary zeros? Explain your reasoning.

In Exercises 69–74, write the expression as a complex number in standard form.

69. \((3 + 4i) - (7 - 5i) + 2i(9 + 12i)\)

70. \(3i(2 + 5i) + (6 - 7i) - (9 + i)\)

71. \((3 + 5i)(2 - 7i)\)

72. \(2i(5 - 12i)\)

73. \((2 + 4i^2) + (1 - 9i^6) - (3 + i^4)\)

74. \((8 - 2i^3) + (3 - 7i^8) - (4 + i^6)\)

75. **OPEN-ENDED** Find two imaginary numbers whose sum and product are real numbers. How are the imaginary numbers related?

76. **COMPARING METHODS** Describe the two different methods shown for writing the complex expression in standard form. Which method do you prefer? Explain.

**Method 1**

\[
4i(2 - 3i) + 4i(1 - 2i) = 8i - 12i^2 + 4i - 8i^2 \\
= 8i - 12(-1) + 4i - 8(-1) \\
= 20 + 12i
\]

**Method 2**

\[
4i(2 - 3i) + 4i(1 - 2i) = 4i[(2 - 3i) + (1 - 2i)] \\
= 4i[3 - 5i] \\
= 12i - 20i^2 \\
= 12i - 20(-1) \\
= 20 + 12i
\]

77. **CRITICAL THINKING** Determine whether each statement is true or false. If it is true, give an example. If it is false, give a counterexample.

a. The sum of two imaginary numbers is an imaginary number.

b. The product of two pure imaginary numbers is a real number.

c. A pure imaginary number is an imaginary number.

d. A complex number is a real number.

78. **THOUGHT PROVOKING** Create a circuit that has an impedance of \(14 - 3i\).

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Determine whether the given value of \( x \) is a solution to the equation. *(Skills Review Handbook)*

79. \(3(x - 2) + 4x - 1 = x - 1; x = 1\)

80. \(x^3 - 6 = 2x^2 + 9 - 3x; x = -5\)

81. \(-x^2 + 4x = \frac{19}{3}x^2; x = -\frac{3}{4}\)

Write a quadratic function in vertex form whose graph is shown. *(Section 2.4)*

82.

83.

84.

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