1.4 Solving Linear Systems

Essential Question   How can you determine the number of solutions of a linear system?

A linear system is *consistent* when it has at least one solution. A linear system is *inconsistent* when it has no solution.
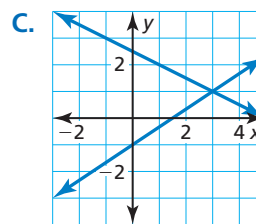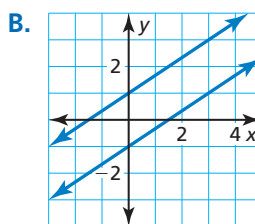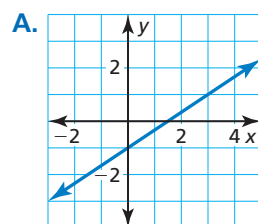
EXPLORATION 1   Recognizing Graphs of Linear Systems

Work with a partner. Match each linear system with its corresponding graph. Explain your reasoning. Then classify the system as *consistent* or *inconsistent*.

a. $2x - 3y = 3$
 $-4x + 6y = 6$

b. $2x - 3y = 3$
 $x + 2y = 5$

c. $2x - 3y = 3$
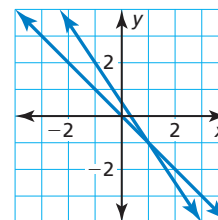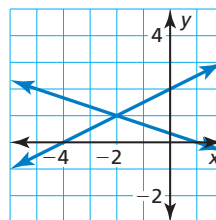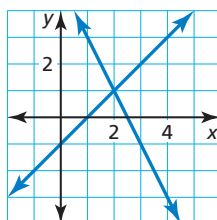$$ $-4x + 6y = -6$

A.

B.

C.

EXPLORATION 2   Solving Systems of Linear Equations

Work with a partner. Solve each linear system by substitution or elimination. Then use the graph of the system below to check your solution.

a. $2x + y = 5$
 $x - y = 1$

b. $x + 3y = 1$
 $-x + 2y = 4$

c. $x + y = 0$
 $3x + 2y = 1$

FINDING AN ENTRY POINT

To be proficient in math, you need to look for entry points to the solution of a problem.

Communicate Your Answer

3. How can you determine the number of solutions of a linear system?

4. Suppose you were given a system of *three* linear equations in *three* variables. Explain how you would approach solving such a system.

5. Apply your strategy in Question 4 to solve the linear system.

$$x + y + z = 1 \qquad \text{Equation 1}$$
$$x - y - z = 3 \qquad \text{Equation 2}$$
$$-x - y + z = -1 \qquad \text{Equation 3}$$

What You Will Learn

▶ Visualize solutions of systems of linear equations in three variables.

▶ Solve systems of linear equations in three variables algebraically.

▶ Solve real-life problems.

Visualizing Solutions of Systems

A **linear equation in three variables** x, y, and z is an equation of the form $ax + by + cz = d$, where a, b, and c are not all zero.

The following is an example of a **system of three linear equations** in three variables.

$$3x + 4y - 8z = -3 \qquad \text{Equation 1}$$
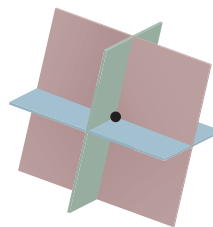$$x + y + 5z = -12 \qquad \text{Equation 2}$$
$$4x - 2y + z = 10 \qquad \text{Equation 3}$$

A **solution** of such a system is an **ordered triple** (x, y, z) whose coordinates make each equation true.

The graph of a linear equation in three variables is a plane in three-dimensional space. The graphs of three such equations that form a system are three planes whose intersection determines the number of solutions of the system, as shown in the diagrams below.

Exactly One Solution
The planes intersect in a single point, which is the solution of the system.

Infinitely Many Solutions
The planes intersect in a line. Every point on the line is a solution of the system.

The planes could also be the same plane. Every point in the plane is a solution of the system.

No Solution
There are no points in common with all three planes.

Solving Systems of Equations Algebraically

The algebraic methods you used to solve systems of linear equations in two variables can be extended to solve a system of linear equations in three variables.

Core Concept

Solving a Three-Variable System

Step 1 Rewrite the linear system in three variables as a linear system in two variables by using the substitution or elimination method.

Step 2 Solve the new linear system for both of its variables.

Step 3 Substitute the values found in Step 2 into one of the original equations and solve for the remaining variable.

When you obtain a false equation, such as $0 = 1$, in any of the steps, the system has no solution.

When you do not obtain a false equation, but obtain an identity such as $0 = 0$, the system has infinitely many solutions.

LOOKING FOR STRUCTURE

The coefficient of -1 in Equation 3 makes y a convenient variable to eliminate.

EXAMPLE 1 **Solving a Three-Variable System (One Solution)**

Solve the system.

$$4x + 2y + 3z = 12 \qquad \text{Equation 1}$$
$$2x - 3y + 5z = -7 \qquad \text{Equation 2}$$
$$6x - y + 4z = -3 \qquad \text{Equation 3}$$

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$$
\begin{array}{ll}
\begin{aligned}
4x + 2y + 3z &= 12 \\
12x - 2y + 8z &= -6 \\
\hline
16x \quad\;\; + 11z &= 6
\end{aligned}
&
\begin{array}{l}
\text{Add 2 times Equation 3 to} \\
\text{Equation 1 (to eliminate } y). \\
\\
\text{New Equation 1}
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\begin{aligned}
2x - 3y + 5z &= -7 \\
-18x + 3y - 12z &= 9 \\
\hline
-16x \quad\quad\; - 7z &= 2
\end{aligned}
&
\begin{array}{l}
\text{Add } -3 \text{ times Equation 3 to} \\
\text{Equation 2 (to eliminate } y). \\
\\
\text{New Equation 2}
\end{array}
\end{array}
$$

ANOTHER WAY

In Step 1, you could also eliminate x to get two equations in y and z, or you could eliminate z to get two equations in x and y.

Step 2 Solve the new linear system for both of its variables.

$$
\begin{array}{ll}
\begin{aligned}
16x + 11z &= 6 \\
-16x - 7z &= 2 \\
\hline
4z &= 8 \\
z &= 2 \\
x &= -1
\end{aligned}
&
\begin{array}{l}
\text{Add new Equation 1} \\
\text{and new Equation 2.} \\
\\
\text{Solve for } z. \\
\text{Substitute into new Equation 1 or 2 to find } x.
\end{array}
\end{array}
$$

Step 3 Substitute $x = -1$ and $z = 2$ into an original equation and solve for y.

$$
\begin{array}{ll}
6x - y + 4z = -3 & \text{Write original Equation 3.} \\
6(-1) - y + 4(2) = -3 & \text{Substitute } -1 \text{ for } x \text{ and 2 for } z. \\
y = 5 & \text{Solve for } y.
\end{array}
$$

▶ The solution is $x = -1$, $y = 5$, and $z = 2$, or the ordered triple $(-1, 5, 2)$. Check this solution in each of the original equations.

EXAMPLE 2 **Solving a Three-Variable System (No Solution)**

Solve the system.

$$x + y + z = 2 \qquad \text{Equation 1}$$
$$5x + 5y + 5z = 3 \qquad \text{Equation 2}$$
$$4x + y - 3z = -6 \qquad \text{Equation 3}$$

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$$
\begin{aligned}
-5x - 5y - 5z &= -10 \\
\underline{5x + 5y + 5z} &= 3 \\
0 &= -7
\end{aligned}
$$

Add -5 times Equation 1 to Equation 2.

▶ Because you obtain a false equation, the original system has no solution.

ANOTHER WAY

Subtracting Equation 2 from Equation 1 gives $z = 0$. After substituting 0 for z in each equation, you can see that each is equivalent to $y = x + 3$.

EXAMPLE 3 **Solving a Three-Variable System (Many Solutions)**

Solve the system.

$$x - y + z = -3 \qquad \text{Equation 1}$$
$$x - y - z = -3 \qquad \text{Equation 2}$$
$$5x - 5y + z = -15 \qquad \text{Equation 3}$$

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$$
\begin{aligned}
x - y + z &= -3 \\
\underline{x - y - z} &= -3 \\
2x - 2y \phantom{{}+ z} &= -6
\end{aligned}
$$

Add Equation 1 to Equation 2 (to eliminate z).

New Equation 2

$$
\begin{aligned}
x - y - z &= -3 \\
\underline{5x - 5y + z} &= -15 \\
6x - 6y \phantom{{}+ z} &= -18
\end{aligned}
$$

Add Equation 2 to Equation 3 (to eliminate z).

New Equation 3

Step 2 Solve the new linear system for both of its variables.

$$
\begin{aligned}
-6x + 6y &= 18 \\
\underline{6x - 6y} &= -18 \\
0 &= 0
\end{aligned}
$$

Add -3 times new Equation 2 to new Equation 3.

Because you obtain the identity $0 = 0$, the system has infinitely many solutions.

Step 3 Describe the solutions of the system using an ordered triple. One way to do this is to solve new Equation 2 for y to obtain $y = x + 3$. Then substitute $x + 3$ for y in original Equation 1 to obtain $z = 0$.

▶ So, any ordered triple of the form $(x, x + 3, 0)$ is a solution of the system.
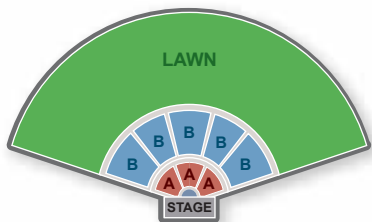
Monitoring Progress 🔊 **Help in English and Spanish at *BigIdeasMath.com***

Solve the system. Check your solution, if possible.

1. $x - 2y + z = -11$
$3x + 2y - z = 7$
$-x + 2y + 4z = -9$

2. $x + y - z = -1$
$4x + 4y - 4z = -2$
$3x + 2y + z = 0$

3. $x + y + z = 8$
$x - y + z = 8$
$2x + y + 2z = 16$

4. In Example 3, describe the solutions of the system using an ordered triple in terms of y.

Solving Real-Life Problems

EXAMPLE 4 **Solving a Multi-Step Problem**

An amphitheater charges $75 for each seat in Section A, $55 for each seat in Section B, and $30 for each lawn seat. There are three times as many seats in Section B as in Section A. The revenue from selling all 23,000 seats is $870,000. How many seats are in each section of the amphitheater?

SOLUTION

Step 1 Write a verbal model for the situation.

$$\boxed{\text{Number of seats in B, } y} = 3 \cdot \boxed{\text{Number of seats in A, } x}$$

$$\boxed{\text{Number of seats in A, } x} + \boxed{\text{Number of seats in B, } y} + \boxed{\text{Number of lawn seats, } z} = \boxed{\text{Total number of seats}}$$

$$75 \cdot \boxed{\text{Number of seats in A, } x} + 55 \cdot \boxed{\text{Number of seats in B, } y} + 30 \cdot \boxed{\text{Number of lawn seats, } z} = \boxed{\text{Total revenue}}$$

Step 2 Write a system of equations.

$y = 3x$	Equation 1
$x + y + z = 23{,}000$	Equation 2
$75x + 55y + 30z = 870{,}000$	Equation 3

Step 3 Rewrite the system in Step 2 as a linear system in *two* variables by substituting $3x$ for y in Equations 2 and 3.

$x + y + z = 23{,}000$	Write Equation 2.
$x + 3x + z = 23{,}000$	Substitute $3x$ for y.
$4x + z = 23{,}000$	New Equation 2
$75x + 55y + 30z = 870{,}000$	Write Equation 3.
$75x + 55(3x) + 30z = 870{,}000$	Substitute $3x$ for y.
$240x + 30z = 870{,}000$	New Equation 3

Step 4 Solve the new linear system for both of its variables.

STUDY TIP

When substituting to find values of other variables, choose original or new equations that are easiest to use.

$-120x - 30z = -690{,}000$	Add -30 times new Equation 2
$\underline{240x + 30z = 870{,}000}$	to new Equation 3.
$120x \qquad = 180{,}000$	
$x = 1500$	Solve for x.
$y = 4500$	Substitute into Equation 1 to find y.
$z = 17{,}000$	Substitute into Equation 2 to find z.

▶ The solution is $x = 1500$, $y = 4500$, and $z = 17{,}000$, or $(1500, 4500, 17{,}000)$. So, there are 1500 seats in Section A, 4500 seats in Section B, and 17,000 lawn seats.

Monitoring Progress 🔊 Help in English and Spanish at *BigIdeasMath.com*

5. **WHAT IF?** On the first day, 10,000 tickets sold, generating $356,000 in revenue. The number of seats sold in Sections A and B are the same. How many lawn seats are still available?

Vocabulary and Core Concept Check

1. **VOCABULARY** The solution of a system of three linear equations is expressed as a(n)_____.

2. **WRITING** Explain how you know when a linear system in three variables has infinitely many solutions.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, solve the system using the elimination method. *(See Example 1.)*

3. $x + y - 2z = 5$
   $-x + 2y + z = 2$
   $2x + 3y - z = 9$

4. $x + 4y - 6z = -1$
   $2x - y + 2z = -7$
   $-x + 2y - 4z = 5$

5. $2x + y - z = 9$
   $-x + 6y + 2z = -17$
   $5x + 7y + z = 4$

6. $3x + 2y - z = 8$
   $-3x + 4y + 5z = -14$
   $x - 3y + 4z = -14$

7. $2x + 2y + 5z = -1$
   $2x - y + z = 2$
   $2x + 4y - 3z = 14$

8. $3x + 2y - 3z = -2$
   $7x - 2y + 5z = -14$
   $2x + 4y + z = 6$

ERROR ANALYSIS In Exercises 9 and 10, describe and correct the error in the first step of solving the system of linear equations.

$$4x - y + 2z = -18$$
$$-x + 2y + z = 11$$
$$3x + 3y - 4z = 44$$

9.
   ✗
   $4x - y + 2z = -18$
   $\underline{-4x + 2y + z = 11}$
   $y + 3z = -7$

10.
   ✗
   $12x - 3y + 6z = -18$
   $\underline{3x + 3y - 4z = 44}$
   $15x + 2z = 26$

In Exercises 11–16, solve the system using the elimination method. *(See Examples 2 and 3.)*

11. $3x - y + 2z = 4$
    $6x - 2y + 4z = -8$
    $2x - y + 3z = 10$

12. $5x + y - z = 6$
    $x + y + z = 2$
    $12x + 4y = 10$

13. $x + 3y - z = 2$
    $x + y - z = 0$
    $3x + 2y - 3z = -1$

14. $x + 2y - z = 3$
    $-2x - y + z = -1$
    $6x - 3y - z = -7$

15. $x + 2y + 3z = 4$
    $-3x + 2y - z = 12$
    $-2x - 2y - 4z = -14$

16. $-2x - 3y + z = -6$
    $x + y - z = 5$
    $7x + 8y - 6z = 31$

17. **MODELING WITH MATHEMATICS** Three orders are placed at a pizza shop. Two small pizzas, a liter of soda, and a salad cost $14; one small pizza, a liter of soda, and three salads cost $15; and three small pizzas, a liter of soda, and two salads cost $22. How much does each item cost?

18. **MODELING WITH MATHEMATICS** Sam's Furniture Store places the following advertisement in the local newspaper. Write a system of equations for the three combinations of furniture. What is the price of each piece of furniture? Explain.

SALE — SAM'S Furniture Store

$1300 Sofa and love seat

$1400 Sofa and two chairs

$1600 Sofa, love seat, and one chair

In Exercises 19–28, solve the system of linear equations using the substitution method. *(See Example 4.)*

19. $-2x + y + 6z = 1$
$3x + 2y + 5z = 16$
$7x + 3y - 4z = 11$

20. $x - 6y - 2z = -8$
$-x + 5y + 3z = 2$
$3x - 2y - 4z = 18$

21. $x + y + z = 4$
$5x + 5y + 5z = 12$
$x - 4y + z = 9$

22. $x + 2y = -1$
$-x + 3y + 2z = -4$
$-x + y - 4z = 10$

23. $2x - 3y + z = 10$
$y + 2z = 13$
$z = 5$

24. $x = 4$
$x + y = -6$
$4x - 3y + 2z = 26$

25. $x + y - z = 4$
$3x + 2y + 4z = 17$
$-x + 5y + z = 8$

26. $2x - y - z = 15$
$4x + 5y + 2z = 10$
$-x - 4y + 3z = -20$

27. $4x + y + 5z = 5$
$8x + 2y + 10z = 10$
$x - y - 2z = -2$

28. $x + 2y - z = 3$
$2x + 4y - 2z = 6$
$-x - 2y + z = -6$

29. PROBLEM SOLVING The number of left-handed people in the world is one-tenth the number of right-handed people. The percent of right-handed people is nine times the percent of left-handed people and ambidextrous people combined. What percent of people are ambidextrous?
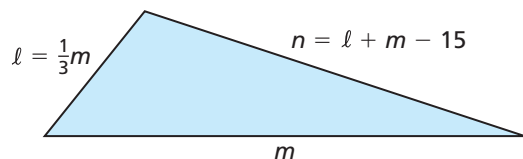

30. MODELING WITH MATHEMATICS Use a system of linear equations to model the data in the following newspaper article. Solve the system to find how many athletes finished in each place.

> Lawrence High prevailed in Saturday's track meet with the help of 20 individual-event placers earning a combined 68 points. A first-place finish earns 5 points, a second-place finish earns 3 points, and a third-place finish earns 1 point. Lawrence had a strong second-place showing, with as many second place finishers as first- and third-place finishers combined.
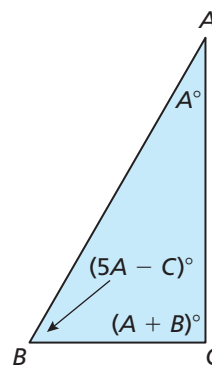
31. WRITING Explain when it might be more convenient to use the elimination method than the substitution method to solve a linear system. Give an example to support your claim.

32. REPEATED REASONING Using what you know about solving linear systems in two and three variables, plan a strategy for how you would solve a system that has *four* linear equations in *four* variables.

MATHEMATICAL CONNECTIONS In Exercises 33 and 34, write and use a linear system to answer the question.

33. The triangle has a perimeter of 65 feet. What are the lengths of sides ℓ, m, and n?

34. What are the measures of angles A, B, and C?

35. OPEN-ENDED Consider the system of linear equations below. Choose nonzero values for a, b, and c so the system satisfies the given condition. Explain your reasoning.
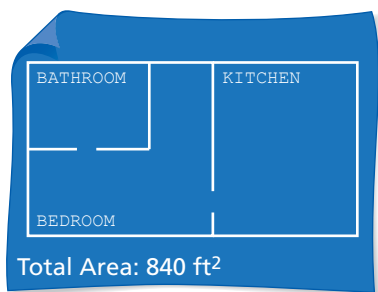
$$x + y + z = 2$$
$$ax + by + cz = 10$$
$$x - 2y + z = 4$$

a. The system has no solution.

b. The system has exactly one solution.

c. The system has infinitely many solutions.

36. MAKING AN ARGUMENT A linear system in three variables has no solution. Your friend concludes that it is not possible for two of the three equations to have any points in common. Is your friend correct? Explain your reasoning.

37. PROBLEM SOLVING A contractor is hired to build an apartment complex. Each 840-square-foot unit has a bedroom, kitchen, and bathroom. The bedroom will be the same size as the kitchen. The owner orders 980 square feet of tile to completely cover the floors of two kitchens and two bathrooms. Determine how many square feet of carpet is needed for each bedroom.
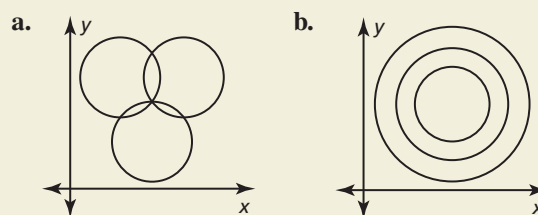

BATHROOM KITCHEN

BEDROOM

Total Area: 840 ft²

38. THOUGHT PROVOKING Does the system of linear equations have more than one solution? Justify your answer.

$$4x + y + z = 0$$
$$2x + \tfrac{1}{2}y - 3z = 0$$
$$-x - \tfrac{1}{4}y - z = 0$$

39. PROBLEM SOLVING A florist must make 5 identical bridesmaid bouquets for a wedding. The budget is $160, and each bouquet must have 12 flowers. Roses cost $2.50 each, lilies cost $4 each, and irises cost $2 each. The florist wants twice as many roses as the other two types of flowers combined.

  a. Write a system of equations to represent this situation, assuming the florist plans to use the maximum budget.

  b. Solve the system to find how many of each type of flower should be in each bouquet.

  c. Suppose there is no limitation on the total cost of the bouquets. Does the problem still have exactly one solution? If so, find the solution. If not, give three possible solutions.

40. HOW DO YOU SEE IT? Determine whether the system of equations that represents the circles has *no solution*, *one solution*, or *infinitely many solutions*. Explain your reasoning.

  a.

  b.

41. CRITICAL THINKING Find the values of a, b, and c so that the linear system shown has $(-1, 2, -3)$ as its only solution. Explain your reasoning.

$$x + 2y - 3z = a$$
$$-x - y + z = b$$
$$2x + 3y - 2z = c$$

42. ANALYZING RELATIONSHIPS Determine which arrangement(s) of the integers -5, 2, and 3 produce a solution of the linear system that consist of only integers. Justify your answer.

$$x - 3y + 6z = 21$$
$$\_\_x + \_\_y + \_\_z = -30$$
$$2x - 5y + 2z = -6$$

43. ABSTRACT REASONING Write a linear system to represent the first three pictures below. Use the system to determine how many tangerines are required to balance the apple in the fourth picture. *Note:* The first picture shows that one tangerine and one apple balance one grapefruit.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Simplify. *(Skills Review Handbook)*

44. $(x - 2)^2$ **45.** $(3m + 1)^2$ **46.** $(2z - 5)^2$ **47.** $(4 - y)^2$

Write a function g described by the given transformation of $f(x) = |x| - 5$. *(Section 1.2)*

48. translation 2 units to the left   **49.** reflection in the x-axis

50. translation 4 units up   **51.** vertical stretch by a factor of 3