1.2 Transformations of Linear and Absolute Value Functions

Essential Question How do the graphs of $y = f(x) + k$, $y = f(x - h)$, and $y = -f(x)$ compare to the graph of the parent function f?

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to visualize results and explore consequences.

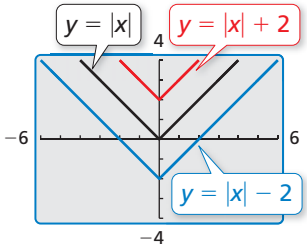
EXPLORATION 1 Transformations of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function

$y = |x| + k$ Transformation

to the graph of the parent function

$f(x) = |x|$. Parent function



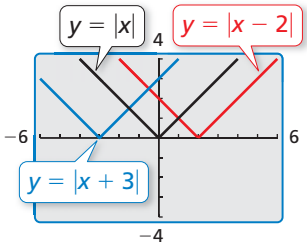
EXPLORATION 2 Transformations of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function

$y = |x - h|$ Transformation

to the graph of the parent function

$f(x) = |x|$. Parent function



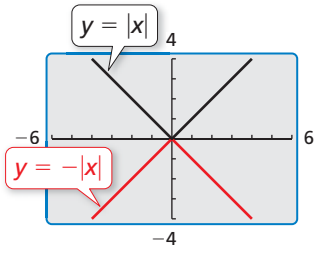
EXPLORATION 3 Transformation of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function

$y = -|x|$ Transformation

to the graph of the parent function

$f(x) = |x|$. Parent function



Communicate Your Answer

4. How do the graphs of $y = f(x) + k$, $y = f(x - h)$, and $y = -f(x)$ compare to the graph of the parent function f?

5. Compare the graph of each function to the graph of its parent function f. Use a graphing calculator to verify your answers are correct.

a. $y = \sqrt{x} - 4$ **b.** $y = \sqrt{x + 4}$ **c.** $y = -\sqrt{x}$

d. $y = x^2 + 1$ **e.** $y = (x - 1)^2$ **f.** $y = -x^2$

1.2 Lesson

What You Will Learn

- Write functions representing translations and reflections.
- Write functions representing stretches and shrinks.
- Write functions representing combinations of transformations.

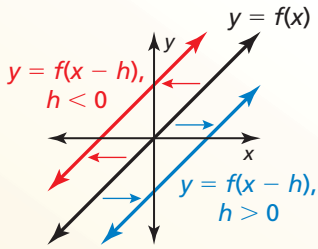
Translations and Reflections

You can use function notation to represent transformations of graphs of functions.

Core Concept

Horizontal Translations

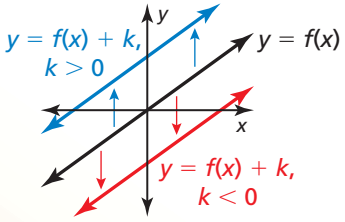
The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$, where $h \neq 0$.



Subtracting h from the **inputs** before evaluating the function shifts the graph left when $h < 0$ and right when $h > 0$.

Vertical Translations

The graph of $y = f(x) + k$ is a vertical translation of the graph of $y = f(x)$, where $k \neq 0$.



Adding k to the **outputs** shifts the graph down when $k < 0$ and up when $k > 0$.

EXAMPLE 1 Writing Translations of Functions

Let $f(x) = 2x + 1$.

a. Write a function g whose graph is a translation 3 units down of the graph of f.

b. Write a function h whose graph is a translation 2 units to the left of the graph of f.

SOLUTION

a. A translation 3 units down is a vertical translation that adds -3 to each output value.

$g(x) = f(x) + (-3)$ Add -3 to the output.

$= 2x + 1 + (-3)$ Substitute $2x + 1$ for $f(x)$.

$= 2x - 2$ Simplify.

▶ The translated function is $g(x) = 2x - 2$.

b. A translation 2 units to the left is a horizontal translation that subtracts –2 from each input value.

$h(x) = f(x - (-2))$ Subtract -2 from the input.

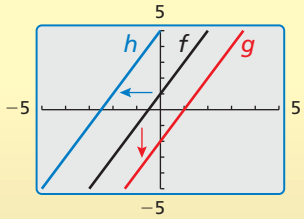
$= f(x + 2)$ Add the opposite.

$= 2(x + 2) + 1$ Replace x with $x + 2$ in $f(x)$.

$= 2x + 5$ Simplify.

▶ The translated function is $h(x) = 2x + 5$.

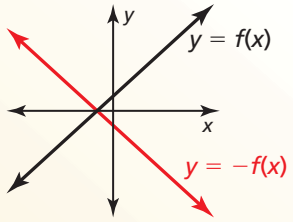
Check



Core Concept

Reflections in the *x*-axis

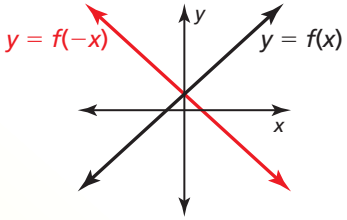
The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.



Multiplying the **outputs** by -1 changes their signs.

Reflections in the *y*-axis

The graph of $y = f(-x)$ is a reflection in the y-axis of the graph of $y = f(x)$.



Multiplying the **inputs** by -1 changes their signs.

STUDY TIP

When you reflect a function in a line, the graphs are symmetric about that line.

EXAMPLE 2 Writing Reflections of Functions

Let $f(x) = |x + 3| + 1$.

a. Write a function g whose graph is a reflection in the x-axis of the graph of f.

b. Write a function h whose graph is a reflection in the y-axis of the graph of f.

SOLUTION

a. A reflection in the x-axis changes the sign of each output value.

$g(x) = -f(x)$ — Multiply the output by -1.

$= -(|x + 3| + 1)$ — Substitute $|x + 3| + 1$ for $f(x)$.

$= -|x + 3| - 1$ — Distributive Property

▶ The reflected function is $g(x) = -|x + 3| - 1$.

b. A reflection in the y-axis changes the sign of each input value.

$h(x) = f(-x)$ — Multiply the input by -1.

$= |-x + 3| + 1$ — Replace x with $-x$ in $f(x)$.

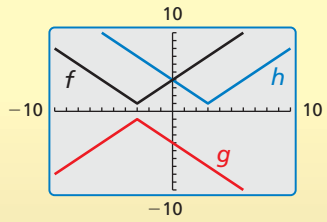
$= |-(x - 3)| + 1$ — Factor out -1.

$= |-1| \cdot |x - 3| + 1$ — Product Property of Absolute Value

$= |x - 3| + 1$ — Simplify.

▶ The reflected function is $h(x) = |x - 3| + 1$.

Check



Monitoring Progress

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Write a function g whose graph represents the indicated transformation of the graph of f. Use a graphing calculator to check your answer.

1. $f(x) = 3x$; translation 5 units up
2. $f(x) = |x| - 3$; translation 4 units to the right
3. $f(x) = -|x + 2| - 1$; reflection in the x-axis
4. $f(x) = \frac{1}{2}x + 1$; reflection in the y-axis

Stretches and Shrinks

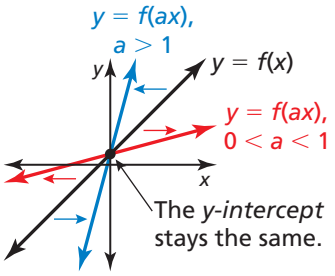
In the previous section, you learned that vertical stretches and shrinks transform graphs. You can also use *horizontal* stretches and shrinks to transform graphs.

Core Concept

Horizontal Stretches and Shrinks

The graph of $y = f(ax)$ is a horizontal stretch or shrink by a factor of $\frac{1}{a}$ of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.

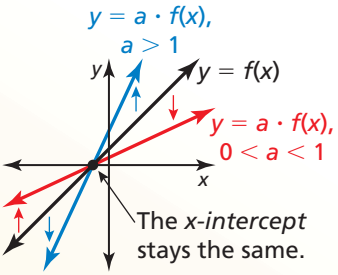
Multiplying the **inputs** by a before evaluating the function stretches the graph horizontally (away from the y-axis) when $0 < a < 1$, and shrinks the graph horizontally (toward the y-axis) when $a > 1$.



Vertical Stretches and Shrinks

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.

Multiplying the **outputs** by a stretches the graph vertically (away from the x-axis) when $a > 1$, and shrinks the graph vertically (toward the x-axis) when $0 < a < 1$.



STUDY TIP

The graphs of $y = f(-ax)$ and $y = -a \cdot f(x)$ represent a stretch or shrink *and* a reflection in the x- or y-axis of the graph of $y = f(x)$.

EXAMPLE 3 Writing Stretches and Shrinks of Functions

Let $f(x) = |x - 3| - 5$. Write (a) a function g whose graph is a horizontal shrink of the graph of f by a factor of $\frac{1}{3}$, and (b) a function h whose graph is a vertical stretch of the graph of f by a factor of 2.

SOLUTION

a. A horizontal shrink by a factor of $\frac{1}{3}$ multiplies each input value by 3.

$g(x) = f(3x)$ Multiply the input by 3.

$= |3x - 3| - 5$ Replace x with $3x$ in $f(x)$.

▶ The transformed function is $g(x) = |3x - 3| - 5$.

b. A vertical stretch by a factor of 2 multiplies each output value by 2.

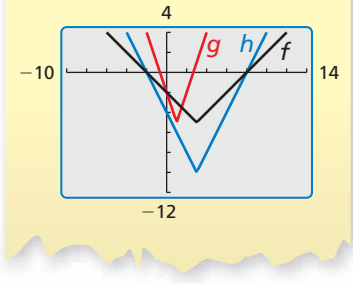
$h(x) = 2 \cdot f(x)$ Multiply the output by 2.

$= 2 \cdot (|x - 3| - 5)$ Substitute $|x - 3| - 5$ for $f(x)$.

$= 2|x - 3| - 10$ Distributive Property

▶ The transformed function is $h(x) = 2|x - 3| - 10$.

Check



Monitoring Progress

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Write a function g whose graph represents the indicated transformation of the graph of f. Use a graphing calculator to check your answer.

5. $f(x) = 4x + 2$; horizontal stretch by a factor of 2

6. $f(x) = |x| - 3$; vertical shrink by a factor of $\frac{1}{3}$

Combinations of Transformations

You can write a function that represents a series of transformations on the graph of another function by applying the transformations one at a time in the stated order.

EXAMPLE 4 Combining Transformations

Let the graph of g be a vertical shrink by a factor of 0.25 followed by a translation 3 units up of the graph of $f(x) = x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the vertical shrink of f.

$h(x) = 0.25 \cdot f(x)$ Multiply the output by 0.25.

$= 0.25x$ Substitute x for $f(x)$.

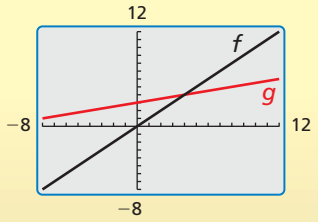
Step 2 Then write a function g that represents the translation of h.

$g(x) = h(x) + 3$ Add 3 to the output.

$= 0.25x + 3$ Substitute $0.25x$ for $h(x)$.

▶ The transformed function is $g(x) = 0.25x + 3$.

Check



EXAMPLE 5 Modeling with Mathematics

You design a computer game. Your revenue for x downloads is given by $f(x) = 2x$. Your profit is \$50 less than 90% of the revenue for x downloads. Describe how to transform the graph of f to model the profit. What is your profit for 100 downloads?

SOLUTION

1. **Understand the Problem** You are given a function that represents your revenue and a verbal statement that represents your profit. You are asked to find the profit for 100 downloads.
2. **Make a Plan** Write a function p that represents your profit. Then use this function to find the profit for 100 downloads.
3. **Solve the Problem** profit = 90% • revenue − 50

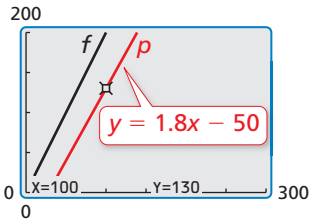
$p(x) = 0.9 \cdot f(x) - 50$

Vertical shrink by a factor of 0.9

Translation 50 units down

$= 0.9 \cdot 2x - 50$ Substitute $2x$ for $f(x)$.

$= 1.8x - 50$ Simplify.



To find the profit for 100 downloads, evaluate p when $x = 100$.

$p(100) = 1.8(100) - 50 = 130$

▶ Your profit is \$130 for 100 downloads.

4. **Look Back** The vertical shrink decreases the slope, and the translation shifts the graph 50 units down. So, the graph of p is below and not as steep as the graph of f.

Monitoring Progress

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7. Let the graph of g be a translation 6 units down followed by a reflection in the x-axis of the graph of $f(x) = |x|$. Write a rule for g. Use a graphing calculator to check your answer.
8. **WHAT IF?** In Example 5, your revenue function is $f(x) = 3x$. How does this affect your profit for 100 downloads?

1.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The function $g(x) = |5x| - 4$ is a horizontal _____ of the function $f(x) = |x| - 4$.

2. **WHICH ONE DOESN'T BELONG?** Which transformation does *not* belong with the other three? Explain your reasoning.

Translate the graph of $f(x) = 2x + 3$ up 2 units.	Shrink the graph of $f(x) = x + 5$ horizontally by a factor of $\frac{1}{2}$.
Stretch the graph of $f(x) = x + 3$ vertically by a factor of 2.	Translate the graph of $f(x) = 2x + 3$ left 1 unit.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write a function g whose graph represents the indicated transformation of the graph of f. Use a graphing calculator to check your answer. *(See Example 1.)*

3. $f(x) = x - 5$; translation 4 units to the left

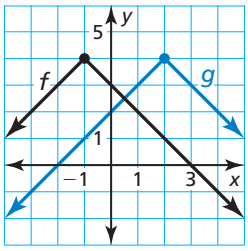
4. $f(x) = x + 2$; translation 2 units to the right

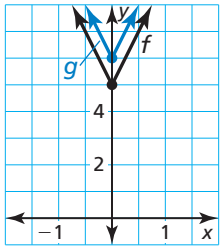
5. $f(x) = |4x + 3| + 2$; translation 2 units down

6. $f(x) = 2x - 9$; translation 6 units up

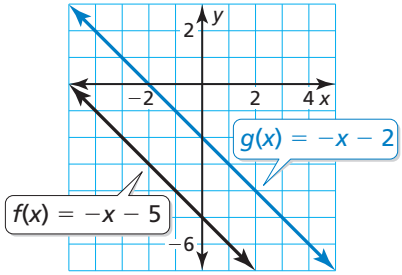
7. $f(x) = 4 - |x + 1|$

8. $f(x) = |4x| + 5$





9. **WRITING** Describe two different translations of the graph of f that result in the graph of g.



10. **PROBLEM SOLVING** You open a café. The function $f(x) = 4000x$ represents your expected net income (in dollars) after being open x weeks. Before you open, you incur an extra expense of \$12,000. What transformation of f is necessary to model this situation? How many weeks will it take to pay off the extra expense?

In Exercises 11–16, write a function g whose graph represents the indicated transformation of the graph of f. Use a graphing calculator to check your answer. *(See Example 2.)*

11. $f(x) = -5x + 2$; reflection in the x-axis

12. $f(x) = \frac{1}{2}x - 3$; reflection in the x-axis

13. $f(x) = |6x| - 2$; reflection in the y-axis

14. $f(x) = |2x - 1| + 3$; reflection in the y-axis

15. $f(x) = -3 + |x - 11|$; reflection in the y-axis

16. $f(x) = -x + 1$; reflection in the y-axis

In Exercises 17–22, write a function g whose graph represents the indicated transformation of the graph of f. Use a graphing calculator to check your answer. *(See Example 3.)*

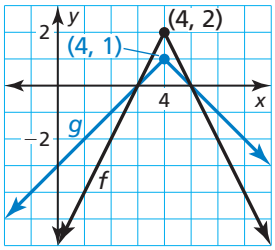
17. $f(x) = x + 2$; vertical stretch by a factor of 5

18. $f(x) = 2x + 6$; vertical shrink by a factor of $\frac{1}{2}$

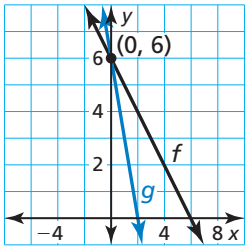
19. $f(x) = |2x| + 4$; horizontal shrink by a factor of $\frac{1}{2}$

20. $f(x) = |x + 3|$; horizontal stretch by a factor of 4

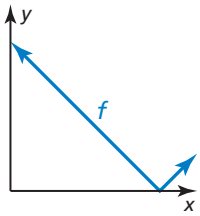
21. $f(x) = -2|x - 4| + 2$



22. $f(x) = 6 - x$



ANALYZING RELATIONSHIPS
In Exercises 23–26, match the graph of the transformation of f with the correct equation shown. Explain your reasoning.



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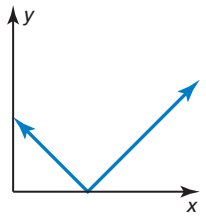

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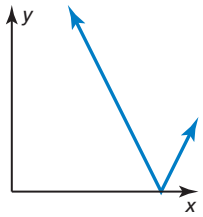

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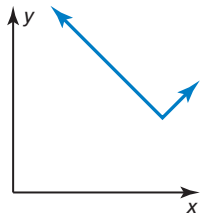

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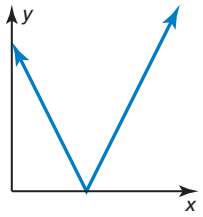


A. $y = 2f(x)$ **B.** $y = f(2x)$
C. $y = f(x + 2)$ **D.** $y = f(x) + 2$

In Exercises 27–32, write a function g whose graph represents the indicated transformations of the graph of f. *(See Example 4.)*

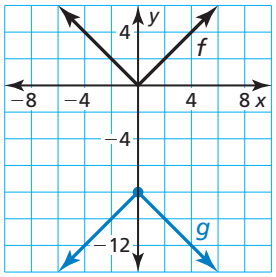
27. $f(x) = x$; vertical stretch by a factor of 2 followed by a translation 1 unit up

28. $f(x) = x$; translation 3 units down followed by a vertical shrink by a factor of $\frac{1}{3}$

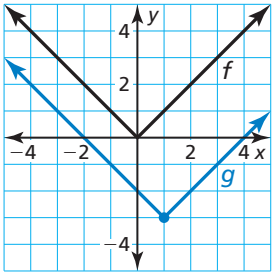
29. $f(x) = |x|$; translation 2 units to the right followed by a horizontal stretch by a factor of 2

30. $f(x) = |x|$; reflection in the y-axis followed by a translation 3 units to the right

31. $f(x) = |x|$



32. $f(x) = |x|$



ERROR ANALYSIS **In Exercises 33 and 34, identify and correct the error in writing the function g whose graph represents the indicated transformations of the graph of f.**

33.

$f(x) = |x|$; translation 3 units to the right followed by a translation 2 units up

$g(x) = |x + 3| + 2$

34.

$f(x) = x$; translation 6 units down followed by a vertical stretch by a factor of 5

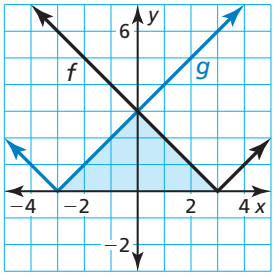
$g(x) = 5x - 6$

35. **MAKING AN ARGUMENT** Your friend claims that when writing a function whose graph represents a combination of transformations, the order is not important. Is your friend correct? Justify your answer.

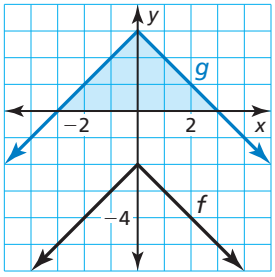
36. MODELING WITH MATHEMATICS During a recent period of time, bookstore sales have been declining. The sales (in billions of dollars) can be modeled by the function $f(t) = -\frac{7}{5}t + 17.2$, where t is the number of years since 2006. Suppose sales decreased at twice the rate. How can you transform the graph of f to model the sales? Explain how the sales in 2010 are affected by this change. *(See Example 5.)*

MATHEMATICAL CONNECTIONS For Exercises 37–40, describe the transformation of the graph of f to the graph of g. Then find the area of the shaded triangle.

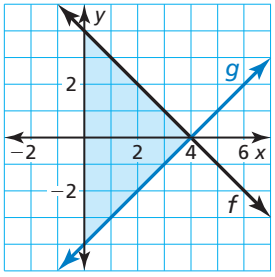
37. $f(x) = |x - 3|$



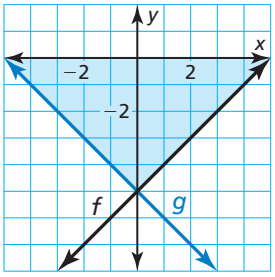
38. $f(x) = -|x| - 2$



39. $f(x) = -x + 4$



40. $f(x) = x - 5$

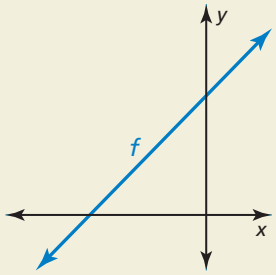


41. ABSTRACT REASONING The functions $f(x) = mx + b$ and $g(x) = mx + c$ represent two parallel lines.

a. Write an expression for the vertical translation of the graph of f to the graph of g.

b. Use the definition of slope to write an expression for the horizontal translation of the graph of f to the graph of g.

42. HOW DO YOU SEE IT? Consider the graph of $f(x) = mx + b$. Describe the effect each transformation has on the slope of the line and the intercepts of the graph.



a. Reflect the graph of f in the y-axis.

b. Shrink the graph of f vertically by a factor of $\frac{1}{3}$.

c. Stretch the graph of f horizontally by a factor of 2.

43. REASONING The graph of $g(x) = -4|x| + 2$ is a reflection in the x-axis, vertical stretch by a factor of 4, and a translation 2 units down of the graph of its parent function. Choose the correct order for the transformations of the graph of the parent function to obtain the graph of g. Explain your reasoning.

44. THOUGHT PROVOKING You are planning a cross-country bicycle trip of 4320 miles. Your distance d (in miles) from the halfway point can be modeled by $d = 72|x - 30|$, where x is the time (in days) and $x = 0$ represents June 1. Your plans are altered so that the model is now a right shift of the original model. Give an example of how this can happen. Sketch both the original model and the shifted model.

45. CRITICAL THINKING Use the correct value 0, −2, or 1 with a, b, and c so the graph of $g(x) = a|x - b| + c$ is a reflection in the x-axis followed by a translation one unit to the left and one unit up of the graph of $f(x) = 2|x - 2| + 1$. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function for the given value of x. *(Skills Review Handbook)*

46. $f(x) = x + 4; x = 3$

47. $f(x) = 4x - 1; x = -1$

48. $f(x) = -x + 3; x = 5$

49. $f(x) = -2x - 2; x = -1$

Create a scatter plot of the data. *(Skills Review Handbook)*

50.

x	8	10	11	12	15
$f(x)$	4	9	10	12	12

51.

x	2	5	6	10	13
$f(x)$	22	13	15	12	6