1.2 Transformations of Linear and Absolute Value Functions

Essential Question How do the graphs of \( y = f(x) + k \), \( y = f(x - h) \), and \( y = -f(x) \) compare to the graph of the parent function \( f \)?

EXPLORATION 1 Transformations of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function \( y = |x| + k \) to the graph of the parent function \( f(x) = |x| \).

EXPLORATION 2 Transformations of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function \( y = |x - h| \) to the graph of the parent function \( f(x) = |x| \).

EXPLORATION 3 Transformation of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function \( y = -|x| \) to the graph of the parent function \( f(x) = |x| \).

Communicate Your Answer

4. How do the graphs of \( y = f(x) + k \), \( y = f(x - h) \), and \( y = -f(x) \) compare to the graph of the parent function \( f \)?

5. Compare the graph of each function to the graph of its parent function \( f \). Use a graphing calculator to verify your answers are correct.

   a. \( y = \sqrt{x} - 4 \)    b. \( y = \sqrt{x} + 4 \)    c. \( y = -\sqrt{x} \)
   d. \( y = x^2 + 1 \)    e. \( y = (x - 1)^2 \)    f. \( y = -x^2 \)
What You Will Learn

- Write functions representing translations and reflections.
- Write functions representing stretches and shrinks.
- Write functions representing combinations of transformations.

Translations and Reflections

You can use function notation to represent transformations of graphs of functions.

Core Concept

**Horizontal Translations**
The graph of \( y = f(x - h) \) is a horizontal translation of the graph of \( y = f(x) \), where \( h \neq 0 \).

### Vertical Translations

The graph of \( y = f(x) + k \) is a vertical translation of the graph of \( y = f(x) \), where \( k \neq 0 \).

**EXAMPLE 1** Writing Translations of Functions

Let \( f(x) = 2x + 1 \).

a. Write a function \( g \) whose graph is a translation 3 units down of the graph of \( f \).

b. Write a function \( h \) whose graph is a translation 2 units to the left of the graph of \( f \).

**SOLUTION**

a. A translation 3 units down is a vertical translation that adds \(-3\) to each output value.

\[
g(x) = f(x) + (-3) \\
= 2x + 1 + (-3) \\
= 2x - 2
\]

The translated function is \( g(x) = 2x - 2 \).

b. A translation 2 units to the left is a horizontal translation that subtracts \(-2\) from each input value.

\[
h(x) = f(x - (-2)) \\
= f(x + 2) \\
= 2(x + 2) + 1 \\
= 2x + 5
\]

The translated function is \( h(x) = 2x + 5 \).
Let \( f(x) = |x + 3| + 1 \).

a. Write a function \( g \) whose graph is a reflection in the \( x \)-axis of the graph of \( f \).

b. Write a function \( h \) whose graph is a reflection in the \( y \)-axis of the graph of \( f \).

**SOLUTION**

a. A reflection in the \( x \)-axis changes the sign of each output value.

\[
g(x) = -f(x)
\]

\[
= -(|x + 3| + 1)
\]

\[
= -|x + 3| - 1
\]

The reflected function is \( g(x) = -|x + 3| - 1 \).

b. A reflection in the \( y \)-axis changes the sign of each input value.

\[
h(x) = f(-x)
\]

\[
= |-x + 3| + 1
\]

\[
= -(x - 3) + 1
\]

\[
= -1 \cdot |x - 3| + 1
\]

\[
= |x - 3| + 1
\]

The reflected function is \( h(x) = |x - 3| + 1 \).

**Monitoring Progress**

Write a function \( g \) whose graph represents the indicated transformation of the graph of \( f \). Use a graphing calculator to check your answer.

1. \( f(x) = 3x \); translation 5 units up
2. \( f(x) = |x| - 3 \); translation 4 units to the right
3. \( f(x) = -|x + 2| - 1 \); reflection in the \( x \)-axis
4. \( f(x) = \frac{1}{2}x + 1 \); reflection in the \( y \)-axis
Stretches and Shrinks

In the previous section, you learned that vertical stretches and shrinks transform graphs. You can also use horizontal stretches and shrinks to transform graphs.

Core Concept

Horizontal Stretches and Shrinks

The graph of \( y = f(ax) \) is a horizontal stretch or shrink by a factor of \( \frac{1}{a} \) of the graph of \( y = f(x) \), where \( a > 0 \) and \( a \neq 1 \).

Multiplying the inputs by \( a \) before evaluating the function stretches the graph horizontally (away from the \( y \)-axis) when \( 0 < a < 1 \), and shrinks the graph horizontally (toward the \( y \)-axis) when \( a > 1 \).

Vertical Stretches and Shrinks

The graph of \( y = a \cdot f(x) \) is a vertical stretch or shrink by a factor of \( a \) of the graph of \( y = f(x) \), where \( a > 0 \) and \( a \neq 1 \).

Multiplying the outputs by \( a \) stretches the graph vertically (away from the \( x \)-axis) when \( a > 1 \), and shrinks the graph vertically (toward the \( x \)-axis) when \( 0 < a < 1 \).

EXAMPLE 3 Writing Stretches and Shrinks of Functions

Let \( f(x) = |x - 3| - 5 \). Write (a) a function \( g \) whose graph is a horizontal shrink of the graph of \( f \) by a factor of \( \frac{1}{3} \) and (b) a function \( h \) whose graph is a vertical stretch of the graph of \( f \) by a factor of 2.

SOLUTION

a. A horizontal shrink by a factor of \( \frac{1}{3} \) multiplies each input value by 3.

\[
g(x) = f(\frac{1}{3}x)
\]

Multiply the input by 3.

\[
g(x) = |\frac{1}{3}x - 3| - 5
\]

Replace \( x \) with \( \frac{1}{3}x \) in \( f(x) \).

The transformed function is \( g(x) = |\frac{1}{3}x - 3| - 5 \).

b. A vertical stretch by a factor of 2 multiplies each output value by 2.

\[
h(x) = 2 \cdot f(x)
\]

Multiply the output by 2.

\[
h(x) = 2 \cdot (|x - 3| - 5)
\]

Substitute \( |x - 3| - 5 \) for \( f(x) \).

\[
h(x) = 2|x - 3| - 10
\]

Distributive Property

The transformed function is \( h(x) = 2|x - 3| - 10 \).

Monitoring Progress

Write a function \( g \) whose graph represents the indicated transformation of the graph of \( f \). Use a graphing calculator to check your answer.

5. \( f(x) = 4x + 2; \) horizontal stretch by a factor of 2

6. \( f(x) = |x| - 3; \) vertical shrink by a factor of \( \frac{1}{3} \)
Combining Transformations

You can write a function that represents a series of transformations on the graph of another function by applying the transformations one at a time in the stated order.

**EXAMPLE 4** Combining Transformations

Let the graph of \( g \) be a vertical shrink by a factor of 0.25 followed by a translation 3 units up of the graph of \( f(x) = x \). Write a rule for \( g \).

**SOLUTION**

Step 1 First write a function \( h \) that represents the vertical shrink of \( f \).

\[
h(x) = 0.25 \cdot f(x)
\]

Multiply the output by 0.25.

\[
h(x) = 0.25x
\]

Substitute \( x \) for \( f(x) \).

Step 2 Then write a function \( g \) that represents the translation of \( h \).

\[
g(x) = h(x) + 3
\]

Add 3 to the output.

\[
g(x) = 0.25x + 3
\]

Substitute 0.25 \( x \) for \( h(x) \).

The transformed function is \( g(x) = 0.25x + 3 \).

**EXAMPLE 5** Modeling with Mathematics

You design a computer game. Your revenue for \( x \) downloads is given by \( f(x) = 2x \). Your profit is $50 less than 90% of the revenue for \( x \) downloads. Describe how to transform the graph of \( f \) to model the profit. What is your profit for 100 downloads?

**SOLUTION**

1. **Understand the Problem** You are given a function that represents your revenue and a verbal statement that represents your profit. You are asked to find the profit for 100 downloads.

2. **Make a Plan** Write a function \( p \) that represents your profit. Then use this function to find the profit for 100 downloads.

3. **Solve the Problem**

\[
profit = 0.9 \cdot revenue - 50
\]

\[
p(x) = 0.9 \cdot f(x) - 50
\]

\[
p(x) = 0.9 \cdot 2x - 50
\]

\[
p(x) = 1.8x - 50
\]

Substitute 2\( x \) for \( f(x) \). Simplify.

To find the profit for 100 downloads, evaluate \( p \) when \( x = 100 \).

\[
p(100) = 1.8(100) - 50 = 130
\]

Your profit is $130 for 100 downloads.

4. **Look Back** The vertical shrink decreases the slope, and the translation shifts the graph 50 units down. So, the graph of \( p \) is below and not as steep as the graph of \( f \).

**Monitoring Progress**

7. Let the graph of \( g \) be a translation 6 units down followed by a reflection in the \( x \)-axis of the graph of \( f(x) = |x| \). Write a rule for \( g \). Use a graphing calculator to check your answer.

8. **WHAT IF?** In Example 5, your revenue function is \( f(x) = 3x \). How does this affect your profit for 100 downloads?
1. **COMPLETE THE SENTENCE** The function \( g(x) = |5x| - 4 \) is a horizontal __________ of the function \( f(x) = |x| - 4 \).

2. **WHICH ONE DOESN’T BELONG?** Which transformation does not belong with the other three? Explain your reasoning.
   - Translate the graph of \( f(x) = 2x + 3 \) up 2 units.
   - Shrink the graph of \( f(x) = x + 3 \) horizontally by a factor of \( \frac{1}{2} \).
   - Stretch the graph of \( f(x) = x + 3 \) vertically by a factor of 2.
   - Translate the graph of \( f(x) = 2x + 3 \) left 1 unit.

**Monitoring Progress and Modeling with Mathematics**

In Exercises 3–8, write a function \( g \) whose graph represents the indicated transformation of the graph of \( f \). Use a graphing calculator to check your answer. (See Example 1.)

3. \( f(x) = x - 5; \) translation 4 units to the left
4. \( f(x) = x + 2; \) translation 2 units to the right
5. \( f(x) = |4x + 3| + 2; \) translation 2 units down
6. \( f(x) = 2x - 9; \) translation 6 units up
7. \( f(x) = 4 - |x + 1| \)
8. \( f(x) = |4x| + 5 \)

9. **WRITING** Describe two different translations of the graph of \( f \) that result in the graph of \( g \).

10. **PROBLEM SOLVING** You open a café. The function \( f(x) = 4000x \) represents your expected net income (in dollars) after being open \( x \) weeks. Before you open, you incur an extra expense of $12,000. What transformation of \( f \) is necessary to model this situation? How many weeks will it take to pay off the extra expense?

In Exercises 11–16, write a function \( g \) whose graph represents the indicated transformation of the graph of \( f \). Use a graphing calculator to check your answer. (See Example 2.)

11. \( f(x) = -5x + 2; \) reflection in the x-axis
12. \( f(x) = \frac{1}{2}x - 3; \) reflection in the x-axis
13. \( f(x) = |6x| - 2; \) reflection in the y-axis
14. \( f(x) = |2x - 1| + 3; \) reflection in the y-axis
15. \( f(x) = -3 + |x - 11|; \) reflection in the y-axis
16. \( f(x) = -x + 1; \) reflection in the y-axis
In Exercises 17–22, write a function \( g \) whose graph represents the indicated transformation of the graph of \( f \). Use a graphing calculator to check your answer. (See Example 3.)

17. \( f(x) = x + 2 \); vertical stretch by a factor of 5
18. \( f(x) = 2x + 6 \); vertical shrink by a factor of \( \frac{1}{2} \)
19. \( f(x) = |2x| + 4 \); horizontal shrink by a factor of \( \frac{1}{2} \)
20. \( f(x) = |x + 3| \); horizontal stretch by a factor of 4
21. \( f(x) = -2|x - 4| + 2 \)

22. \( f(x) = 6 - x \)

In Exercises 27–32, write a function \( g \) whose graph represents the indicated transformations of the graph of \( f \). (See Example 4.)

27. \( f(x) = x \); vertical stretch by a factor of 2 followed by a translation 1 unit up
28. \( f(x) = x \); translation 3 units down followed by a vertical shrink by a factor of \( \frac{1}{3} \)
29. \( f(x) = |x| \); translation 2 units to the right followed by a horizontal stretch by a factor of 2
30. \( f(x) = |x| \); reflection in the \( y \)-axis followed by a translation 3 units to the right
31. \( f(x) = |x| \)
32. \( f(x) = |x| \)

ERROR ANALYSIS In Exercises 33 and 34, identify and correct the error in writing the function \( g \) whose graph represents the indicated transformations of the graph of \( f \).

33. \( f(x) = |x| \); translation 3 units to the right followed by a translation 2 units up
\[ g(x) = |x + 3| + 2 \]

34. \( f(x) = x \); translation 6 units down followed by a vertical stretch by a factor of 5
\[ g(x) = 5x - 6 \]

35. MAKING AN ARGUMENT Your friend claims that when writing a function whose graph represents a combination of transformations, the order is not important. Is your friend correct? Justify your answer.
36. **MODELING WITH MATHEMATICS** During a recent period of time, bookstore sales have been declining. The sales (in billions of dollars) can be modeled by the function \( f(t) = -\frac{1}{2}t + 17.2 \), where \( t \) is the number of years since 2006. Suppose sales decreased at twice the rate. How can you transform the graph of \( f \) to model the sales? Explain how the sales in 2010 are affected by this change. (See Example 5.)

**MATHEMATICAL CONNECTIONS** For Exercises 37–40, describe the transformation of the graph of \( f \) to the graph of \( g \). Then find the area of the shaded triangle.

37. \( f(x) = |x - 3| \) \hspace{1cm} 38. \( f(x) = -|x| - 2 \)

39. \( f(x) = -x + 4 \) \hspace{1cm} 40. \( f(x) = x - 5 \)

41. **ABSTRACT REASONING** The functions \( f(x) = mx + b \) and \( g(x) = mx + c \) represent two parallel lines.
   a. Write an expression for the vertical translation of the graph of \( f \) to the graph of \( g \).
   b. Use the definition of slope to write an expression for the horizontal translation of the graph of \( f \) to the graph of \( g \).

42. **HOW DO YOU SEE IT?** Consider the graph of \( f(x) = mx + b \). Describe the effect each transformation has on the slope of the line and the intercepts of the graph.

43. **REASONING** The graph of \( g(x) = -4|x| + 2 \) is a reflection in the \( x \)-axis, vertical stretch by a factor of 4, and a translation 2 units down of the graph of its parent function. Choose the correct order for the transformations of the graph of the parent function to obtain the graph of \( g \). Explain your reasoning.

44. **THOUGHT PROVOKING** You are planning a cross-country bicycle trip of 4320 miles. Your distance \( d \) (in miles) from the halfway point can be modeled by \( d = 72|x - 30| \), where \( x \) is the time (in days) and \( x = 0 \) represents June 1. Your plans are altered so that the model is now a right shift of the original model. Give an example of how this can happen. Sketch both the original model and the shifted model.

45. **CRITICAL THINKING** Use the correct value 0, -2, or 1 with \( a, b, \) and \( c \) so the graph of \( g(x) = a|x - b| + c \) is a reflection in the \( x \)-axis followed by a translation one unit to the left and one unit up of the graph of \( f(x) = 2|x - 2| + 1 \). Explain your reasoning.

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**Maintaining Mathematical Proficiency** Reviewing what you learned in previous grades and lessons

Evaluate the function for the given value of \( x \). (Skills Review Handbook)

46. \( f(x) = x + 4; x = 3 \) \hspace{1cm} 47. \( f(x) = 4x - 1; x = -1 \) \hspace{1cm} 48. \( f(x) = -x + 3; x = 5 \) \hspace{1cm} 49. \( f(x) = -2x - 2; x = -1 \)

Create a scatter plot of the data. (Skills Review Handbook)

50. \[
\begin{array}{c|cccccc}
\hline
x & 8 & 10 & 11 & 12 & 15 \\
\hline
f(x) & 4 & 9 & 10 & 12 & 12 \\
\hline
\end{array}
\]

51. \[
\begin{array}{c|cccccc}
\hline
x & 2 & 5 & 6 & 10 & 13 \\
\hline
f(x) & 22 & 13 & 15 & 12 & 6 \\
\hline
\end{array}
\]