8.4 Graphing $f(x) = a(x - h)^2 + k$

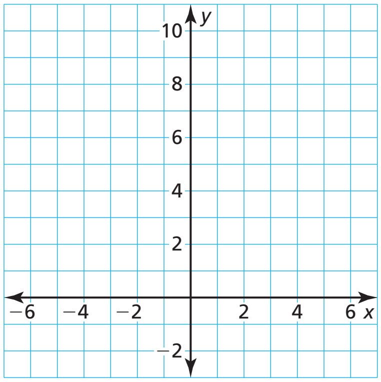
Essential Question

How can you describe the graph of $f(x) = a(x - h)^2$?

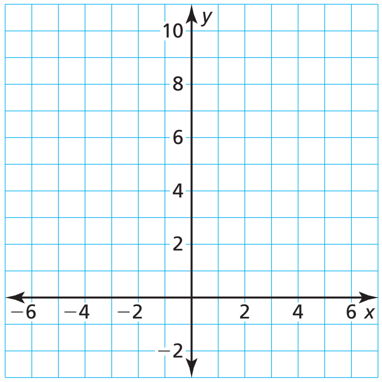
EXPLORATION 1 Graphing $y = a(x - h)^2$ When $h > 0$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. How does the value of h affect the graph of $y = a(x - h)^2$?

a. $f(x) = x^2$ and $g(x) = (x - 2)^2$



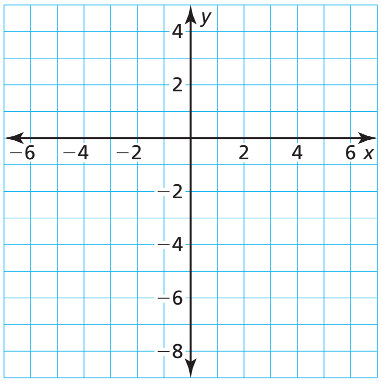
b. $f(x) = 2x^2$ and $g(x) = 2(x - 2)^2$



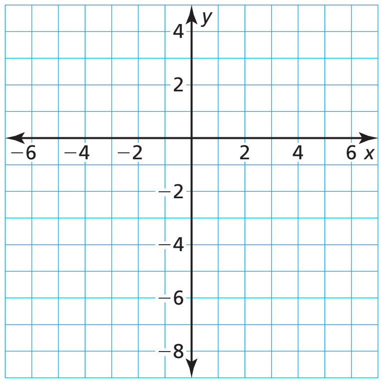
EXPLORATION 2 Graphing $y = a(x - h)^2$ When $h < 0$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. How does the value of h affect the graph of $y = a(x - h)^2$?

a. $f(x) = -x^2$ and $g(x) = -(x + 2)^2$



b. $f(x) = -2x^2$ and $g(x) = -2(x + 2)^2$



USING TOOLS STRATEGICALLY

To be proficient in math, you need to consider the available tools, such as a graphing calculator, when solving a mathematical problem.

Communicate Your Answer

3. How can you describe the graph of $f(x) = a(x - h)^2$?

4. Without graphing, describe the graph of each function. Use a graphing calculator to check your answer.

a. $y = (x - 3)^2$

b. $y = (x + 3)^2$

c. $y = -(x - 3)^2$

8.4 Lesson

What You Will Learn

- Identify even and odd functions.
- Graph quadratic functions of the form $f(x) = a(x - h)^2$.
- Graph quadratic functions of the form $f(x) = a(x - h)^2 + k$.
- Model real-life problems using $f(x) = a(x - h)^2 + k$.

Core Vocabulary

even function, *p. 442*
odd function, *p. 442*
vertex form (of a quadratic function), *p. 444*

Previous
reflection

Identifying Even and Odd Functions

Core Concept

Even and Odd Functions

A function $y = f(x)$ is **even** when $f(-x) = f(x)$ for each x in the domain of f. The graph of an even function is symmetric about the y-axis.

A function $y = f(x)$ is **odd** when $f(-x) = -f(x)$ for each x in the domain of f. The graph of an odd function is symmetric about the origin. A graph is *symmetric about the origin* when it looks the same after reflections in the x-axis and then in the y-axis.

STUDY TIP
The graph of an odd function looks the same after a 180° rotation about the origin.

EXAMPLE 1 Identifying Even and Odd Functions

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = 2x$ **b.** $g(x) = x^2 - 2$ **c.** $h(x) = 2x^2 + x - 2$

SOLUTION

a.
$f(x) = 2x$ — Write the original function.
$f(-x) = 2(-x)$ — Substitute $-x$ for x.
$= -2x$ — Simplify.
$= -f(x)$ — Substitute $f(x)$ for $2x$.

▶ Because $f(-x) = -f(x)$, the function is odd.

b.
$g(x) = x^2 - 2$ — Write the original function.
$g(-x) = (-x)^2 - 2$ — Substitute $-x$ for x.
$= x^2 - 2$ — Simplify.
$= g(x)$ — Substitute $g(x)$ for $x^2 - 2$.

▶ Because $g(-x) = g(x)$, the function is even.

c.
$h(x) = 2x^2 + x - 2$ — Write the original function.
$h(-x) = 2(-x)^2 + (-x) - 2$ — Substitute $-x$ for x.
$= 2x^2 - x - 2$ — Simplify.

▶ Because $h(x) = 2x^2 + x - 2$ and $-h(x) = -2x^2 - x + 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

STUDY TIP
Most functions are neither even nor odd.

Monitoring Progress

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Determine whether the function is *even*, *odd*, or *neither*.

1. $f(x) = 5x$ **2.** $g(x) = 2^x$ **3.** $h(x) = 2x^2 + 3$

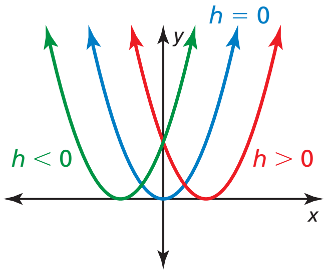
Graphing $f(x) = a(x - h)^2$

Core Concept

Graphing $f(x) = a(x - h)^2$

- When $h > 0$, the graph of $f(x) = a(x - h)^2$ is a horizontal translation h units right of the graph of $f(x) = ax^2$.
- When $h < 0$, the graph of $f(x) = a(x - h)^2$ is a horizontal translation $|h|$ units left of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = a(x - h)^2$ is $(h, 0)$, and the axis of symmetry is $x = h$.



EXAMPLE 2 **Graphing $y = a(x - h)^2$**

Graph $g(x) = \frac{1}{2}(x - 4)^2$. Compare the graph to the graph of $f(x) = x^2$.

SOLUTION

Step 1 Graph the axis of symmetry. Because $h = 4$, graph $x = 4$.

Step 2 Plot the vertex. Because $h = 4$, plot $(4, 0)$.

Step 3 Find and plot two more points on the graph. Choose two x-values less than the x-coordinate of the vertex. Then find $g(x)$ for each x-value.

When $x = 0$:

$g(0) = \frac{1}{2}(0 - 4)^2$

$= 8$

When $x = 2$:

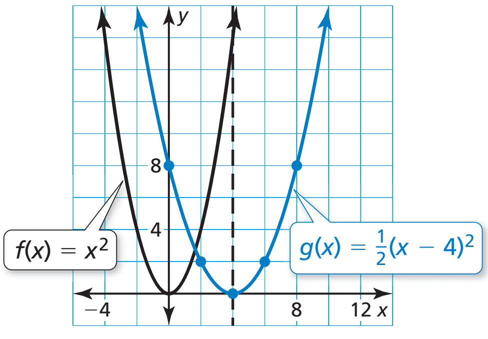
$g(2) = \frac{1}{2}(2 - 4)^2$

$= 2$

So, plot $(0, 8)$ and $(2, 2)$.

Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot $(8, 8)$ and $(6, 2)$.

Step 5 Draw a smooth curve through the points.



▶ Both graphs open up. The graph of g is wider than the graph of f. The axis of symmetry $x = 4$ and the vertex $(4, 0)$ of the graph of g are 4 units right of the axis of symmetry $x = 0$ and the vertex $(0, 0)$ of the graph of f. So, the graph of g is a translation 4 units right and a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f.

ANOTHER WAY

In Step 3, you could instead choose two x-values greater than the x-coordinate of the vertex.

STUDY TIP

From the graph, you can see that $f(x) = x^2$ is an even function. However, $g(x) = \frac{1}{2}(x - 4)^2$ is neither even nor odd.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

4. $g(x) = 2(x + 5)^2$

5. $h(x) = -(x - 2)^2$

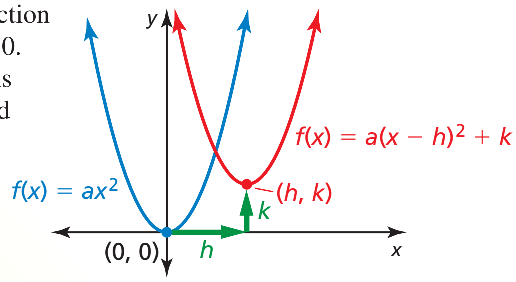
Graphing $f(x) = a(x - h)^2 + k$

Core Concept

Graphing $f(x) = a(x - h)^2 + k$

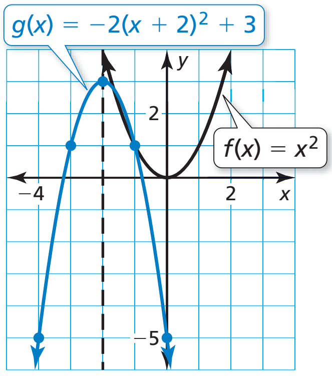
The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The graph of $f(x) = a(x - h)^2 + k$ is a translation h units horizontally and k units vertically of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = a(x - h)^2 + k$ is (h, k), and the axis of symmetry is $x = h$.



EXAMPLE 3 Graphing $y = a(x - h)^2 + k$

Graph $g(x) = -2(x + 2)^2 + 3$. Compare the graph to the graph of $f(x) = x^2$.



SOLUTION

Step 1 Graph the axis of symmetry. Because $h = -2$, graph $x = -2$.

Step 2 Plot the vertex. Because $h = -2$ and $k = 3$, plot $(-2, 3)$.

Step 3 Find and plot two more points on the graph. Choose two x-values less than the x-coordinate of the vertex. Then find $g(x)$ for each x-value. So, plot $(-4, -5)$ and $(-3, 1)$.

x	-4	-3
$g(x)$	-5	1

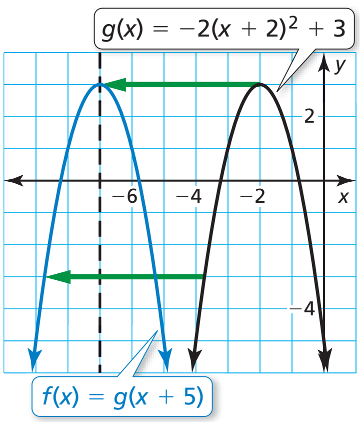
Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot $(-1, 1)$ and $(0, -5)$.

Step 5 Draw a smooth curve through the points.

▶ The graph of g opens down and is narrower than the graph of f. The vertex of the graph of g, $(-2, 3)$, is 2 units left and 3 units up of the vertex of the graph of f, $(0, 0)$. So, the graph of g is a vertical stretch by a factor of 2, a reflection in the x-axis, and a translation 2 units left and 3 units up of the graph of f.

EXAMPLE 4 Transforming the Graph of $y = a(x - h)^2 + k$

Consider function g in Example 3. Graph $f(x) = g(x + 5)$.



SOLUTION

The function f is of the form $y = g(x - h)$, where $h = -5$. So, the graph of f is a horizontal translation 5 units left of the graph of g. To graph f, subtract 5 from the x-coordinates of the points on the graph of g.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

6. $g(x) = 3(x - 1)^2 + 6$

7. $h(x) = \frac{1}{2}(x + 4)^2 - 2$

8. Consider function g in Example 3. Graph $f(x) = g(x) - 3$.

Modeling Real-Life Problems

EXAMPLE 5 Modeling with Mathematics

Water fountains are usually designed to give a specific visual effect. For example, the water fountain shown consists of streams of water that are shaped like parabolas. Notice how the streams are designed to land on the underwater spotlights. Write and graph a quadratic function that models the path of a stream of water with a maximum height of 5 feet, represented by a vertex of (3, 5), landing on a spotlight 6 feet from the water jet, represented by (6, 0).

SOLUTION

1. **Understand the Problem** You know the vertex and another point on the graph that represents the parabolic path. You are asked to write and graph a quadratic function that models the path.

2. **Make a Plan** Use the given points and the vertex form to write a quadratic function. Then graph the function.

3. **Solve the Problem**
Use the vertex form, vertex (3, 5), and point (6, 0) to find the value of a.

$f(x) = a(x - h)^2 + k$	Write the vertex form of a quadratic function.
$f(x) = a(x - 3)^2 + 5$	Substitute 3 for h and 5 for k.
$0 = a(6 - 3)^2 + 5$	Substitute 6 for x and 0 for $f(x)$.
$0 = 9a + 5$	Simplify.
$-\frac{5}{9} = a$	Solve for a.

So, $f(x) = -\frac{5}{9}(x - 3)^2 + 5$ models the path of a stream of water. Now graph the function.

Step 1 Graph the axis of symmetry. Because $h = 3$, graph $x = 3$.

Step 2 Plot the vertex, (3, 5).

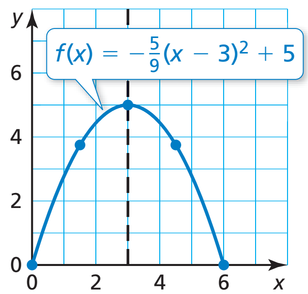
Step 3 Find and plot two more points on the graph. Because the x-axis represents the water surface, the graph should only contain points with nonnegative values of $f(x)$. You know that (6, 0) is on the graph. To find another point, choose an x-value between $x = 3$ and $x = 6$. Then find the corresponding value of $f(x)$.

$$f(4.5) = -\frac{5}{9}(4.5 - 3)^2 + 5 = 3.75$$

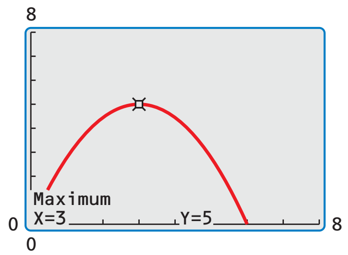
So, plot (6, 0) and (4.5, 3.75).

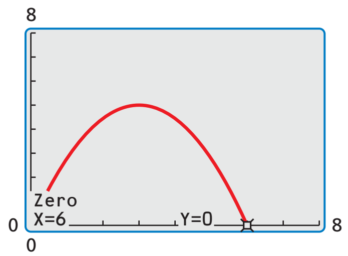
Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot (0, 0) and (1.5, 3.75).

Step 5 Draw a smooth curve through the points.



4. **Look Back** Use a graphing calculator to graph $f(x) = -\frac{5}{9}(x - 3)^2 + 5$. Use the *maximum* feature to verify that the maximum value is 5. Then use the *zero* feature to verify that $x = 6$ is a zero of the function.





Monitoring Progress

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9. **WHAT IF?** The vertex is (3, 6). Write and graph a quadratic function that models the path.

8.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Compare the graph of an even function with the graph of an odd function.

2. **OPEN-ENDED** Write a quadratic function whose graph has a vertex of (1, 2).

3. **WRITING** Describe the transformation from the graph of $f(x) = ax^2$ to the graph of $g(x) = a(x - h)^2 + k$.

4. **WHICH ONE DOESN'T BELONG?** Which function does *not* belong with the other three? Explain your reasoning.

$f(x) = 8(x + 4)^2$	$f(x) = (x - 2)^2 + 4$	$f(x) = 2(x + 0)^2$	$f(x) = 3(x + 1)^2 + 1$

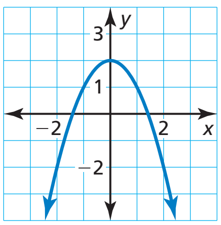
Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, determine whether the function is *even*, *odd*, or *neither*. *(See Example 1.)*

5. $f(x) = 4x + 3$
6. $g(x) = 3x^2$
7. $h(x) = 5^x + 2$
8. $m(x) = 2x^2 - 7x$
9. $p(x) = -x^2 + 8$
10. $f(x) = -\frac{1}{2}x$
11. $n(x) = 2x^2 - 7x + 3$
12. $r(x) = -6x^2 + 5$

In Exercises 13–18, determine whether the function represented by the graph is *even*, *odd*, or *neither*.

13.



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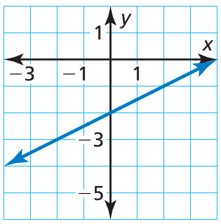

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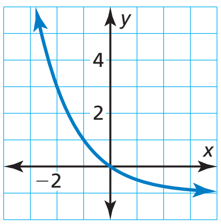

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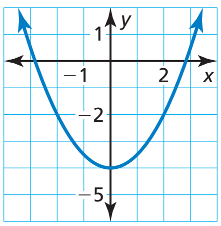

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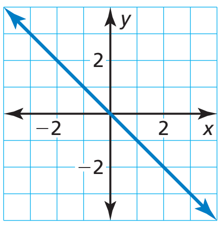
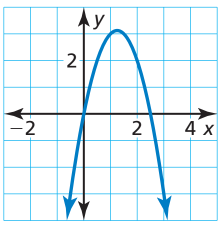


18.



In Exercises 19–22, find the vertex and the axis of symmetry of the graph of the function.

19. $f(x) = 3(x + 1)^2$
20. $f(x) = \frac{1}{4}(x - 6)^2$
21. $y = -\frac{1}{8}(x - 4)^2$
22. $y = -5(x + 9)^2$

In Exercises 23–28, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 2.)*

23. $g(x) = 2(x + 3)^2$
24. $p(x) = 3(x - 1)^2$
25. $r(x) = \frac{1}{4}(x + 10)^2$
26. $n(x) = \frac{1}{3}(x - 6)^2$
27. $d(x) = \frac{1}{5}(x - 5)^2$
28. $q(x) = 6(x + 2)^2$

29. **ERROR ANALYSIS** Describe and correct the error in determining whether the function $f(x) = x^2 + 3$ is even, odd, or neither.

✗

$f(x) = x^2 + 3$

$f(-x) = (-x)^2 + 3$

$= x^2 + 3$

$= f(x)$

So, $f(x)$ is an odd function.

30. **ERROR ANALYSIS** Describe and correct the error in finding the vertex of the graph of the function.

✗

$y = -(x + 8)^2$

Because $h = -8$, the vertex is $(0, -8)$.

In Exercises 31–34, find the vertex and the axis of symmetry of the graph of the function.

31. $y = -6(x + 4)^2 - 3$

32. $f(x) = 3(x - 3)^2 + 6$

33. $f(x) = -4(x + 3)^2 + 1$

34. $y = -(x - 6)^2 - 5$

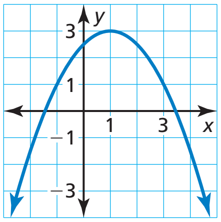
In Exercises 35–38, match the function with its graph.

35. $y = -(x + 1)^2 - 3$

36. $y = -\frac{1}{2}(x - 1)^2 + 3$

37. $y = \frac{1}{3}(x - 1)^2 + 3$

38. $y = 2(x + 1)^2 - 3$

A.


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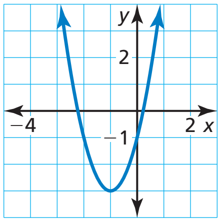

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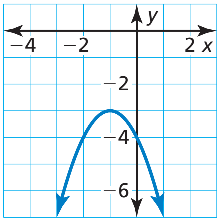

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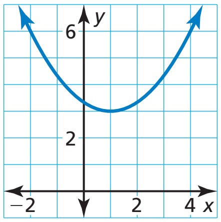


In Exercises 39–44, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 3.)*

39. $h(x) = (x - 2)^2 + 4$

40. $g(x) = (x + 1)^2 - 7$

41. $r(x) = 4(x - 1)^2 - 5$

42. $n(x) = -(x + 4)^2 + 2$

43. $g(x) = -\frac{1}{3}(x + 3)^2 - 2$

44. $r(x) = \frac{1}{2}(x - 2)^2 - 4$

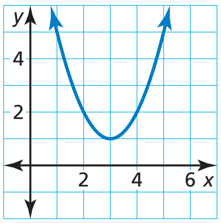
In Exercises 45–48, let $f(x) = (x - 2)^2 + 1$. Match the function with its graph.

45. $g(x) = f(x - 1)$

46. $r(x) = f(x + 2)$

47. $h(x) = f(x) + 2$

48. $p(x) = f(x) - 3$

A.


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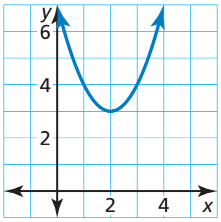

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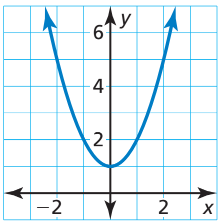

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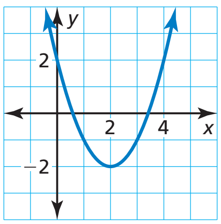


In Exercises 49–54, graph *g*. *(See Example 4.)*

49. $f(x) = 2(x - 1)^2 + 1;\ g(x) = f(x + 3)$

50. $f(x) = -(x + 1)^2 + 2;\ g(x) = \frac{1}{2}f(x)$

51. $f(x) = -3(x + 5)^2 - 6;\ g(x) = 2f(x)$

52. $f(x) = 5(x - 3)^2 - 1;\ g(x) = f(x) - 6$

53. $f(x) = (x + 3)^2 + 5;\ g(x) = f(x - 4)$

54. $f(x) = -2(x - 4)^2 - 8;\ g(x) = -f(x)$

55. MODELING WITH MATHEMATICS The height (in meters) of a bird diving to catch a fish is represented by $h(t) = 5(t - 2.5)^2$, where t is the number of seconds after beginning the dive.

a. Graph h.

b. Another bird's dive is represented by $r(t) = 2h(t)$. Graph r.

c. Compare the graphs. Which bird starts its dive from a greater height? Explain.

56. MODELING WITH MATHEMATICS A kicker punts a football. The height (in yards) of the football is represented by $f(x) = -\frac{1}{9}(x - 30)^2 + 25$, where x is the horizontal distance (in yards) from the kicker's goal line.

a. Graph f. Describe the domain and range.

b. On the next possession, the kicker punts the football. The height of the football is represented by $g(x) = f(x + 5)$. Graph g. Describe the domain and range.

c. Compare the graphs. On which possession does the kicker punt closer to his goal line? Explain.

In Exercises 57–62, write a quadratic function in vertex form whose graph has the given vertex and passes through the given point.

57. vertex: (1, 2); passes through (3, 10)

58. vertex: (−3, 5); passes through (0, −14)

59. vertex: (−2, −4); passes through (−1, −6)

60. vertex: (1, 8); passes through (3, 12)

61. vertex: (5, −2); passes through (7, 0)

62. vertex: (−5, −1); passes through (−2, 2)

63. MODELING WITH MATHEMATICS A portion of a roller coaster track is in the shape of a parabola. Write and graph a quadratic function that models this portion of the roller coaster with a maximum height of 90 feet, represented by a vertex of (25, 90), passing through the point (50, 0). *(See Example 5.)*

64. MODELING WITH MATHEMATICS A flare is launched from a boat and travels in a parabolic path until reaching the water. Write and graph a quadratic function that models the path of the flare with a maximum height of 300 meters, represented by a vertex of (59, 300), landing in the water at the point (119, 0).

In Exercises 65–68, rewrite the quadratic function in vertex form.

65. $y = 2x^2 - 8x + 4$

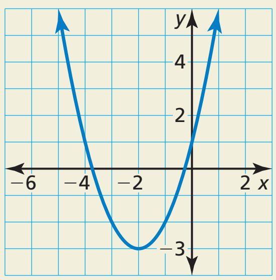
66. $y = 3x^2 + 6x - 1$

67. $f(x) = -5x^2 + 10x + 3$

68. $f(x) = -x^2 - 4x + 2$

69. REASONING Can a function be symmetric about the x-axis? Explain.

70. HOW DO YOU SEE IT? The graph of a quadratic function is shown. Determine which symbols to use to complete the vertex form of the quadratic function. Explain your reasoning.



$y = a(x \;\square\; 2)^2 \;\square\; 3$

In Exercises 71–74, describe the transformation from the graph of f to the graph of h. Write an equation that represents h in terms of x.

71. $f(x) = -(x + 1)^2 - 2$
$h(x) = f(x) + 4$

72. $f(x) = 2(x - 1)^2 + 1$
$h(x) = f(x - 5)$

73. $f(x) = 4(x - 2)^2 + 3$
$h(x) = 2f(x)$

74. $f(x) = -(x + 5)^2 - 6$
$h(x) = \frac{1}{3}f(x)$

75. REASONING The graph of $y = x^2$ is translated 2 units right and 5 units down. Write an equation for the function in vertex form and in standard form. Describe advantages of writing the function in each form.

76. THOUGHT PROVOKING Which of the following are true? Justify your answers.

a. Any constant multiple of an even function is even.

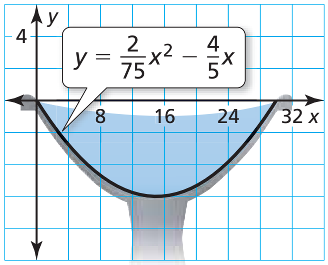
b. Any constant multiple of an odd function is odd.

c. The sum or difference of two even functions is even.

d. The sum or difference of two odd functions is odd.

e. The sum or difference of an even function and an odd function is odd.

77. COMPARING FUNCTIONS A cross section of a birdbath can be modeled by $y = \frac{1}{81}(x - 18)^2 - 4$, where x and y are measured in inches. The graph shows the cross section of another birdbath.



a. Which birdbath is deeper? Explain.

b. Which birdbath is wider? Explain.

78. REASONING Compare the graphs of $y = 2x^2 + 8x + 8$ and $y = x^2$ without graphing the functions. How can factoring help you compare the parabolas? Explain.

79. MAKING AN ARGUMENT Your friend says all absolute value functions are even because of their symmetry. Is your friend correct? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. *(Section 7.4)*

80. $x(x - 1) = 0$

81. $(x + 3)(x - 8) = 0$

82. $(3x - 9)(4x + 12) = 0$