8.3 Graphing $f(x) = ax^2 + bx + c$

Essential Question How can you find the vertex of the graph of $f(x) = ax^2 + bx + c$?

EXPLORATION 1 Comparing x-Intercepts with the Vertex

Work with a partner.

a. Sketch the graphs of $y = 2x^2 - 8x$ and $y = 2x^2 - 8x + 6$.

b. What do you notice about the x-coordinate of the vertex of each graph?

c. Use the graph of $y = 2x^2 - 8x$ to find its x-intercepts. Verify your answer by solving $0 = 2x^2 - 8x$.

d. Compare the value of the x-coordinate of the vertex with the values of the x-intercepts.

EXPLORATION 2 Finding x-Intercepts

Work with a partner.

a. Solve $0 = ax^2 + bx$ for x by factoring.

b. What are the x-intercepts of the graph of $y = ax^2 + bx$?

c. Copy and complete the table to verify your answer.

x	$y = ax^2 + bx$
0	
$-\frac{b}{a}$	

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements.

EXPLORATION 3 Deductive Reasoning

Work with a partner. Complete the following logical argument.

The x-intercepts of the graph of $y = ax^2 + bx$ are 0 and $-\frac{b}{a}$.

The vertex of the graph of $y = ax^2 + bx$ occurs when $x = $ ______.

The vertices of the graphs of $y = ax^2 + bx$ and $y = ax^2 + bx + c$ have the same x-coordinate.

The vertex of the graph of $y = ax^2 + bx + c$ occurs when $x = $ ______.

Communicate Your Answer

4. How can you find the vertex of the graph of $f(x) = ax^2 + bx + c$?

5. Without graphing, find the vertex of the graph of $f(x) = x^2 - 4x + 3$. Check your result by graphing.

8.3 Lesson

Core Vocabulary

maximum value, *p. 433*
minimum value, *p. 433*

Previous
independent variable
dependent variable

What You Will Learn

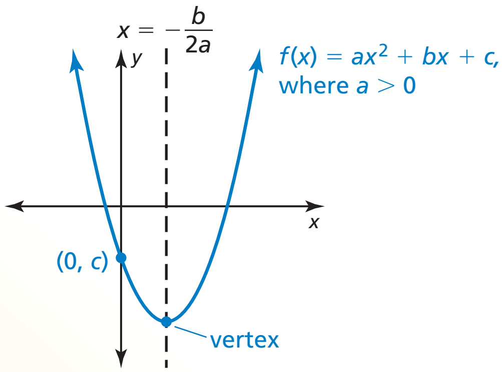
- Graph quadratic functions of the form $f(x) = ax^2 + bx + c$.
- Find maximum and minimum values of quadratic functions.

Graphing $f(x) = ax^2 + bx + c$

Core Concept

Graphing $f(x) = ax^2 + bx + c$

- The graph opens up when $a > 0$, and the graph opens down when $a < 0$.
- The y-intercept is c.
- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- The axis of symmetry is $x = -\frac{b}{2a}$.



EXAMPLE 1 **Finding the Axis of Symmetry and the Vertex**

Find (a) the axis of symmetry and (b) the vertex of the graph of $f(x) = 2x^2 + 8x - 1$.

SOLUTION

a. Find the axis of symmetry when $a = 2$ and $b = 8$.

$x = -\frac{b}{2a}$ — Write the equation for the axis of symmetry.

$x = -\frac{8}{2(2)}$ — Substitute 2 for a and 8 for b.

$x = -2$ — Simplify.

▶ The axis of symmetry is $x = -2$.

b. The axis of symmetry is $x = -2$, so the x-coordinate of the vertex is -2. Use the function to find the y-coordinate of the vertex.

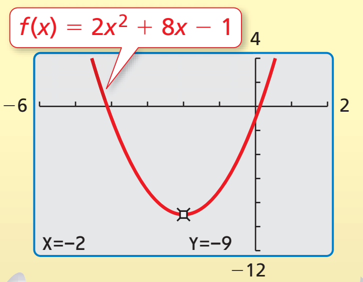
$f(x) = 2x^2 + 8x - 1$ — Write the function.

$f(-2) = 2(-2)^2 + 8(-2) - 1$ — Substitute -2 for x.

$= -9$ — Simplify.

▶ The vertex is $(-2, -9)$.

Check



Monitoring Progress

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Find (a) the axis of symmetry and (b) the vertex of the graph of the function.

1. $f(x) = 3x^2 - 2x$ **2.** $g(x) = x^2 + 6x + 5$ **3.** $h(x) = -\frac{1}{2}x^2 + 7x - 4$

EXAMPLE 2 Graphing $f(x) = ax^2 + bx + c$

Graph $f(x) = 3x^2 - 6x + 5$. Describe the domain and range.

SOLUTION

Step 1 Find and graph the axis of symmetry.

$$x = -\frac{b}{2a} = -\frac{(-6)}{2(3)} = 1 \qquad \text{Substitute and simplify.}$$

COMMON ERROR

Be sure to include the negative sign before the fraction when finding the axis of symmetry.

Step 2 Find and plot the vertex.

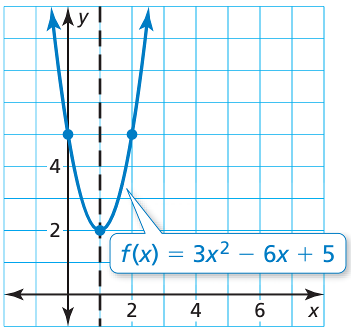
The axis of symmetry is $x = 1$, so the x-coordinate of the vertex is 1. Use the function to find the y-coordinate of the vertex.

$f(x) = 3x^2 - 6x + 5$ Write the function.

$f(1) = 3(1)^2 - 6(1) + 5$ Substitute 1 for x.

$= 2$ Simplify.

So, the vertex is (1, 2).



Step 3 Use the y-intercept to find two more points on the graph.

Because $c = 5$, the y-intercept is 5. So, (0, 5) lies on the graph. Because the axis of symmetry is $x = 1$, the point (2, 5) also lies on the graph.

Step 4 Draw a smooth curve through the points.

▶ The domain is all real numbers. The range is $y \geq 2$.

REMEMBER

The domain is the set of all possible input values of the independent variable x. The range is the set of all possible output values of the dependent variable y.

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Graph the function. Describe the domain and range.

4. $h(x) = 2x^2 + 4x + 1$ **5.** $k(x) = x^2 - 8x + 7$ **6.** $p(x) = -5x^2 - 10x - 2$

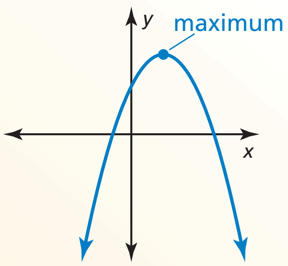
Finding Maximum and Minimum Values

Core Concept

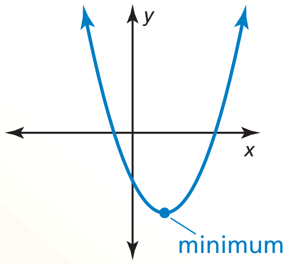
Maximum and Minimum Values

The y-coordinate of the vertex of the graph of $f(x) = ax^2 + bx + c$ is the **maximum value** of the function when $a < 0$ or the **minimum value** of the function when $a > 0$.

$f(x) = ax^2 + bx + c, a < 0$



$f(x) = ax^2 + bx + c, a > 0$



EXAMPLE 3 Finding a Maximum or Minimum Value

Tell whether the function $f(x) = -4x^2 - 24x - 19$ has a minimum value or a maximum value. Then find the value.

SOLUTION

For $f(x) = -4x^2 - 24x - 19$, $a = -4$ and $-4 < 0$. So, the parabola opens down and the function has a maximum value. To find the maximum value, find the y-coordinate of the vertex.

First, find the x-coordinate of the vertex. Use $a = -4$ and $b = -24$.

$$x = -\frac{b}{2a} = -\frac{(-24)}{2(-4)} = -3 \qquad \text{Substitute and simplify.}$$

Then evaluate the function when $x = -3$ to find the y-coordinate of the vertex.

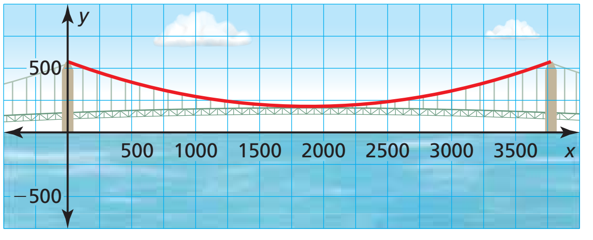
$$f(-3) = -4(-3)^2 - 24(-3) - 19 \qquad \text{Substitute } -3 \text{ for } x.$$

$$= 17 \qquad \text{Simplify.}$$

▶ The maximum value is 17.

EXAMPLE 4 Finding a Minimum Value

The suspension cables between the two towers of the Mackinac Bridge in Michigan form a parabola that can be modeled by $y = 0.000098x^2 - 0.37x + 552$, where x and y are measured in feet. What is the height of the cable above the water at its lowest point?



SOLUTION

The lowest point of the cable is at the vertex of the parabola. Find the x-coordinate of the vertex. Use $a = 0.000098$ and $b = -0.37$.

$$x = -\frac{b}{2a} = -\frac{(-0.37)}{2(0.000098)} \approx 1888 \qquad \text{Substitute and use a calculator.}$$

Substitute 1888 for x in the equation to find the y-coordinate of the vertex.

$$y = 0.000098(1888)^2 - 0.37(1888) + 552 \approx 203$$

▶ The cable is about 203 feet above the water at its lowest point.

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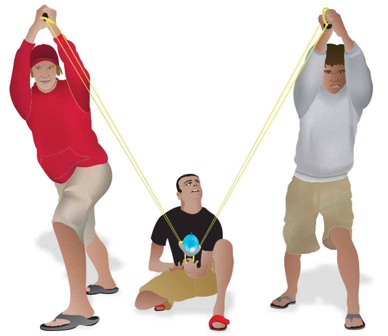
Tell whether the function has a minimum value or a maximum value. Then find the value.

7. $g(x) = 8x^2 - 8x + 6$

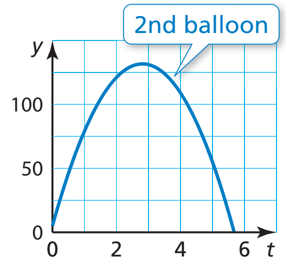
8. $h(x) = -\frac{1}{4}x^2 + 3x + 1$

9. The cables between the two towers of the Tacoma Narrows Bridge in Washington form a parabola that can be modeled by $y = 0.00016x^2 - 0.46x + 507$, where x and y are measured in feet. What is the height of the cable above the water at its lowest point?

EXAMPLE 5 Modeling with Mathematics



A group of friends is launching water balloons. The function $f(t) = -16t^2 + 80t + 5$ represents the height (in feet) of the first water balloon t seconds after it is launched. The height of the second water balloon t seconds after it is launched is shown in the graph. Which water balloon went higher?



SOLUTION

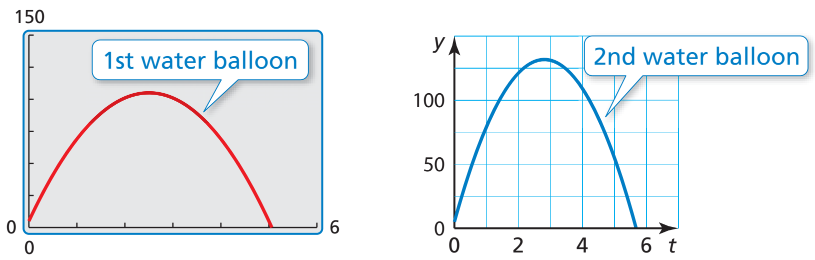
1. **Understand the Problem** You are given a function that represents the height of the first water balloon. The height of the second water balloon is represented graphically. You need to find and compare the maximum heights of the water balloons.

2. **Make a Plan** To compare the maximum heights, represent both functions graphically. Use a graphing calculator to graph $f(t) = -16t^2 + 80t + 5$ in an appropriate viewing window. Then visually compare the heights of the water balloons.

MODELING WITH MATHEMATICS

Because time cannot be negative, use only nonnegative values of t.

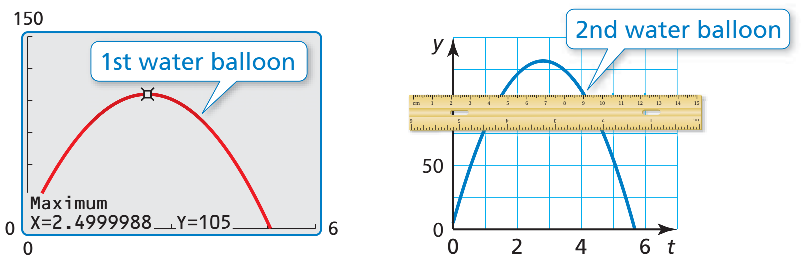
3. **Solve the Problem** Enter the function $f(t) = -16t^2 + 80t + 5$ into your calculator and graph it. Compare the graphs to determine which function has a greater maximum value.



You can see that the second water balloon reaches a height of about 125 feet, while the first water balloon reaches a height of only about 100 feet.

▶ So, the second water balloon went higher.

4. **Look Back** Use the *maximum* feature to determine that the maximum value of $f(t) = -16t^2 + 80t + 5$ is 105. Use a straightedge to represent a height of 105 feet on the graph that represents the second water balloon to clearly see that the second water balloon went higher.



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10. Which balloon is in the air longer? Explain your reasoning.

11. Which balloon reaches its maximum height faster? Explain your reasoning.

8.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Explain how you can tell whether a quadratic function has a maximum value or a minimum value without graphing the function.

2. **DIFFERENT WORDS, SAME QUESTION** Consider the quadratic function $f(x) = -2x^2 + 8x + 24$. Which is different? Find "both" answers.

What is the maximum value of the function?

What is the greatest number in the range of the function?

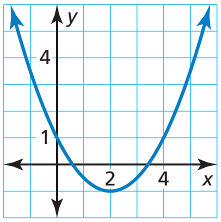
What is the y-coordinate of the vertex of the graph of the function?

What is the axis of symmetry of the graph of the function?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the vertex, the axis of symmetry, and the y-intercept of the graph.

3.



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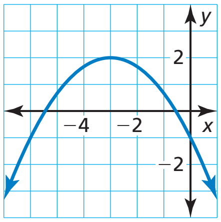

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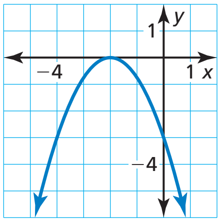

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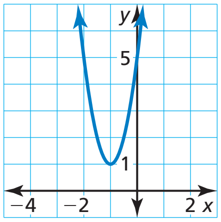


In Exercises 7–12, find (a) the axis of symmetry and (b) the vertex of the graph of the function. *(See Example 1.)*

7. $f(x) = 2x^2 - 4x$

8. $y = 3x^2 + 2x$

9. $y = -9x^2 - 18x - 1$

10. $f(x) = -6x^2 + 24x - 20$

11. $f(x) = \frac{2}{5}x^2 - 4x + 14$

12. $y = -\frac{3}{4}x^2 + 9x - 18$

In Exercises 13–18, graph the function. Describe the domain and range. *(See Example 2.)*

13. $f(x) = 2x^2 + 12x + 4$

14. $y = 4x^2 + 24x + 13$

15. $y = -8x^2 - 16x - 9$

16. $f(x) = -5x^2 + 20x - 7$

17. $y = \frac{2}{3}x^2 - 6x + 5$

18. $f(x) = -\frac{1}{2}x^2 - 3x - 4$

19. **ERROR ANALYSIS** Describe and correct the error in finding the axis of symmetry of the graph of $y = 3x^2 - 12x + 11$.

$$x = -\frac{b}{2a} = \frac{-12}{2(3)} = -2$$

The axis of symmetry is $x = -2$.

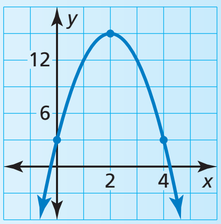
20. **ERROR ANALYSIS** Describe and correct the error in graphing the function $f(x) = x^2 + 4x + 3$.

The axis of symmetry is $x = \frac{b}{2a} = \frac{4}{2(1)} = 2$.

$f(2) = 2^2 + 4(2) + 3 = 15$

So, the vertex is $(2, 15)$.

The y-intercept is 3. So, the points $(0, 3)$ and $(4, 3)$ lie on the graph.



In Exercises 21–26, tell whether the function has a minimum value or a maximum value. Then find the value. *(See Example 3.)*

21. $y = 3x^2 - 18x + 15$

22. $f(x) = -5x^2 + 10x + 7$

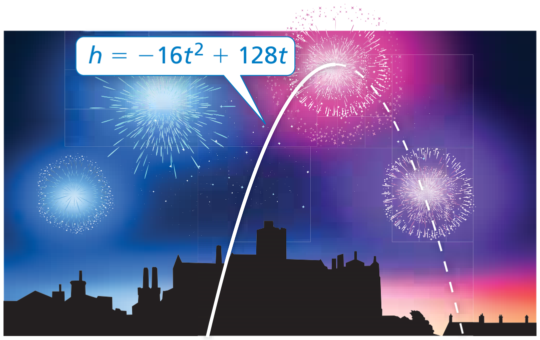
23. $f(x) = -4x^2 + 4x - 2$

24. $y = 2x^2 - 10x + 13$

25. $y = -\frac{1}{2}x^2 - 11x + 6$

26. $f(x) = \frac{1}{5}x^2 - 5x + 27$

27. **MODELING WITH MATHEMATICS** The function shown represents the height h (in feet) of a firework t seconds after it is launched. The firework explodes at its highest point. *(See Example 4.)*



a. When does the firework explode?

b. At what height does the firework explode?

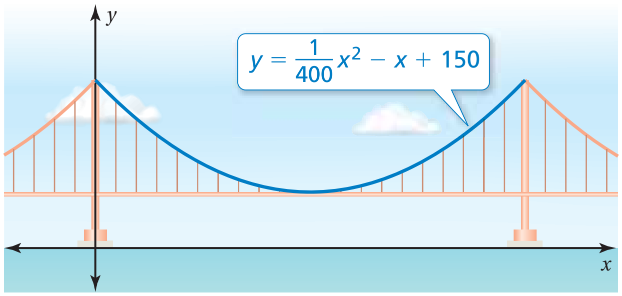
28. **MODELING WITH MATHEMATICS** The function $h(t) = -16t^2 + 16t$ represents the height (in feet) of a horse t seconds after it jumps during a steeplechase.

a. When does the horse reach its maximum height?

b. Can the horse clear a fence that is 3.5 feet tall? If so, by how much?

c. How long is the horse in the air?

29. **MODELING WITH MATHEMATICS** The cable between two towers of a suspension bridge can be modeled by the function shown, where x and y are measured in feet. The cable is at road level midway between the towers.



a. How far from each tower shown is the lowest point of the cable?

b. How high is the road above the water?

c. Describe the domain and range of the function shown.

30. **REASONING** Find the axis of symmetry of the graph of the equation $y = ax^2 + bx + c$ when $b = 0$. Can you find the axis of symmetry when $a = 0$? Explain.

31. **ATTENDING TO PRECISION** The vertex of a parabola is $(3, -1)$. One point on the parabola is $(6, 8)$. Find another point on the parabola. Justify your answer.

32. **MAKING AN ARGUMENT** Your friend claims that it is possible to draw a parabola through any two points with different x-coordinates. Is your friend correct? Explain.

USING TOOLS In Exercises 33–36, use the *minimum* or *maximum* feature of a graphing calculator to approximate the vertex of the graph of the function.

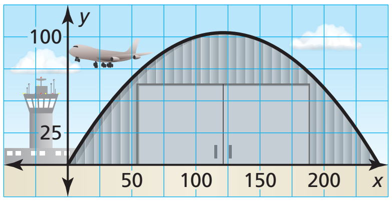
33. $y = 0.5x^2 + \sqrt{2}x - 3$

34. $y = -6.2x^2 + 4.8x - 1$

35. $y = -\pi x^2 + 3x$

36. $y = 0.25x^2 - 5^{2/3}x + 2$

37. **MODELING WITH MATHEMATICS** The opening of one aircraft hangar is a parabolic arch that can be modeled by the equation $y = -0.006x^2 + 1.5x$, where x and y are measured in feet. The opening of a second aircraft hangar is shown in the graph. *(See Example 5.)*



a. Which aircraft hangar is taller?

b. Which aircraft hangar is wider?

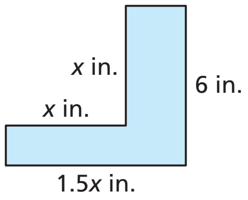
38. **MODELING WITH MATHEMATICS** An office supply store sells about 80 graphing calculators per month for \$120 each. For each \$6 decrease in price, the store expects to sell eight more calculators. The revenue from calculator sales is given by the function $R(n) = (\text{unit price})(\text{units sold})$, or $R(n) = (120 - 6n)(80 + 8n)$, where n is the number of \$6 price decreases.

a. How much should the store charge to maximize monthly revenue?

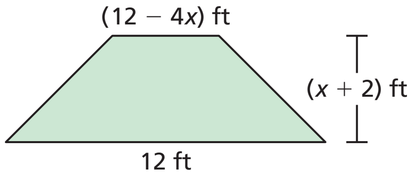
b. Using a different revenue model, the store expects to sell five more calculators for each \$4 decrease in price. Which revenue model results in a greater maximum monthly revenue? Explain.

MATHEMATICAL CONNECTIONS **In Exercises 39 and 40, (a) find the value of x that maximizes the area of the figure and (b) find the maximum area.**

39.

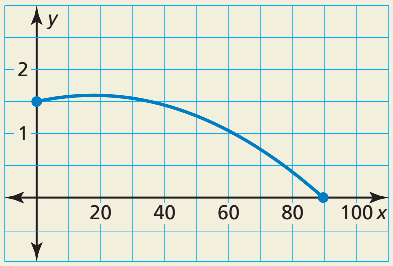


40.



41. **WRITING** Compare the graph of $g(x) = x^2 + 4x + 1$ with the graph of $h(x) = x^2 - 4x + 1$.

42. **HOW DO YOU SEE IT?** During an archery competition, an archer shoots an arrow. The arrow follows the parabolic path shown, where x and y are measured in meters.



a. What is the initial height of the arrow?

b. Estimate the maximum height of the arrow.

c. How far does the arrow travel?

43. **USING TOOLS** The graph of a quadratic function passes through (3, 2), (4, 7), and (9, 2). Does the graph open up or down? Explain your reasoning.

44. **REASONING** For a quadratic function f, what does $f\left(-\frac{b}{2a}\right)$ represent? Explain your reasoning.

45. **PROBLEM SOLVING** Write a function of the form $y = ax^2 + bx$ whose graph contains the points (1, 6) and (3, 6).

46. **CRITICAL THINKING** Parabolas A and B contain the points shown. Identify characteristics of each parabola, if possible. Explain your reasoning.

Parabola A

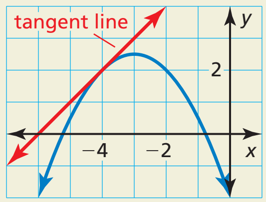
x	y
2	3
6	4

Parabola B

x	y
1	4
3	−4
5	4

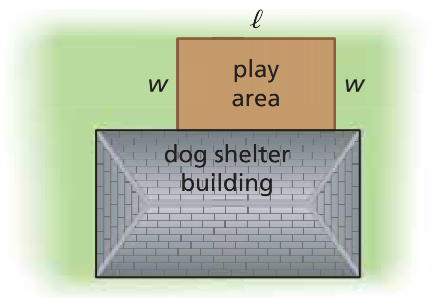
47. **MODELING WITH MATHEMATICS** At a basketball game, an air cannon launches T-shirts into the crowd. The function $y = -\frac{1}{8}x^2 + 4x$ represents the path of a T-shirt. The function $3y = 2x - 14$ represents the height of the bleachers. In both functions, y represents vertical height (in feet) and x represents horizontal distance (in feet). At what height does the T-shirt land in the bleachers?

48. **THOUGHT PROVOKING** One of two classic problems in calculus is finding the slope of a *tangent line* to a curve. An example of a tangent line, which just touches the parabola at one point, is shown.



Approximate the slope of the tangent line to the graph of $y = x^2$ at the point (1, 1). Explain your reasoning.

49. **PROBLEM SOLVING** The owners of a dog shelter want to enclose a rectangular play area on the side of their building. They have k feet of fencing. What is the maximum area of the outside enclosure in terms of k? (*Hint:* Find the y-coordinate of the vertex of the graph of the area function.)



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Describe the transformation(s) from the graph of $f(x) = |x|$ to the graph of the given function. *(Section 3.7)*

50. $q(x) = |x + 6|$ **51.** $h(x) = -0.5|x|$ **52.** $g(x) = |x - 2| + 5$ **53.** $p(x) = 3|x + 1|$