6.6 Geometric Sequences

Essential Question

How can you use a geometric sequence to describe a pattern?

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**.

EXPLORATION 1 Describing Calculator Patterns

Work with a partner. Enter the keystrokes on a calculator and record the results in the table. Describe the pattern.

a. **Step 1** [2] [=]

Step 2 [×] [2] [=]

Step 3 [×] [2] [=]

Step 4 [×] [2] [=]

Step 5 [×] [2] [=]

Step	1	2	3	4	5
Calculator display					

b. **Step 1** [6] [4] [=]

Step 2 [×] [.] [5] [=]

Step 3 [×] [.] [5] [=]

Step 4 [×] [.] [5] [=]

Step 5 [×] [.] [5] [=]

Step	1	2	3	4	5
Calculator display					

c. Use a calculator to make your own sequence. Start with any number and multiply by 3 each time. Record your results in the table.

Step	1	2	3	4	5
Calculator display					

d. Part (a) involves a geometric sequence with a common ratio of 2. What is the common ratio in part (b)? part (c)?

LOOKING FOR REGULARITY IN REPEATED REASONING

To be proficient in math, you need to notice when calculations are repeated and look both for general methods and for shortcuts.

EXPLORATION 2 Folding a Sheet of Paper

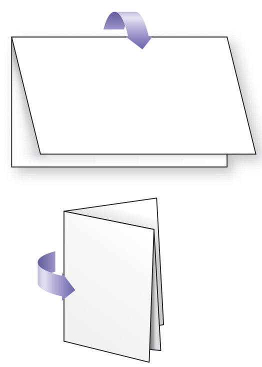
Work with a partner. A sheet of paper is about 0.1 millimeter thick.

a. How thick will it be when you fold it in half once? twice? three times?

b. What is the greatest number of times you can fold a piece of paper in half? How thick is the result?

c. Do you agree with the statement below? Explain your reasoning.

"If it were possible to fold the paper in half 15 times, it would be taller than you."



Communicate Your Answer

3. How can you use a geometric sequence to describe a pattern?

4. Give an example of a geometric sequence from real life other than paper folding.

6.6 Lesson

Core Vocabulary

geometric sequence, *p. 332*
common ratio, *p. 332*

Previous
arithmetic sequence
common difference

What You Will Learn

- Identify geometric sequences.
- Extend and graph geometric sequences.
- Write geometric sequences as functions.

Identifying Geometric Sequences

Core Concept

Geometric Sequence

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**. Each term is found by multiplying the previous term by the common ratio.

$$1, \underbrace{\ }_{\times 5} 5, \underbrace{\ }_{\times 5} 25, \underbrace{\ }_{\times 5} 125, \ldots$$

Terms of a geometric sequence

common ratio

EXAMPLE 1 Identifying Geometric Sequences

Decide whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

a. 120, 60, 30, 15, . . .   **b.** 2, 6, 11, 17, . . .

SOLUTION

a. Find the ratio between each pair of consecutive terms.

$$120 \quad 60 \quad 30 \quad 15$$

$$\frac{60}{120} = \frac{1}{2} \qquad \frac{30}{60} = \frac{1}{2} \qquad \frac{15}{30} = \frac{1}{2}$$

The ratios are the same. The common ratio is $\frac{1}{2}$.

▶ So, the sequence is geometric.

b. Find the ratio between each pair of consecutive terms.

$$2 \quad 6 \quad 11 \quad 17$$

$$\frac{6}{2} = 3 \qquad \frac{11}{6} = 1\frac{5}{6} \qquad \frac{17}{11} = 1\frac{6}{11}$$

There is no common ratio, so the sequence is *not* geometric.

Find the difference between each pair of consecutive terms.

$$2 \quad 6 \quad 11 \quad 17$$

$$6 - 2 = 4 \qquad 11 - 6 = 5 \qquad 17 - 11 = 6$$

There is no common difference, so the sequence is *not* arithmetic.

▶ So, the sequence is *neither* geometric nor arithmetic.

Monitoring Progress

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Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

1. 5, 1, −3, −7, . . .   **2.** 1024, 128, 16, 2, . . .   **3.** 2, 6, 10, 16, . . .

Extending and Graphing Geometric Sequences

EXAMPLE 2 Extending Geometric Sequences

Write the next three terms of each geometric sequence.

a. 3, 6, 12, 24, . . . **b.** 64, −16, 4, −1, . . .

SOLUTION

Use tables to organize the terms and extend each sequence.

a.

Position	1	2	3	4	5	6	7
Term	3	6	12	24	48	96	192

$\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$

Each term is twice the previous term. So, the common ratio is 2.

Multiply a term by 2 to find the next term.

▶ The next three terms are 48, 96, and 192.

b.

Position	1	2	3	4	5	6	7
Term	64	−16	4	−1	$\frac{1}{4}$	$-\frac{1}{16}$	$\frac{1}{64}$

$\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$

Multiply a term by $-\frac{1}{4}$ to find the next term.

▶ The next three terms are $\frac{1}{4}$, $-\frac{1}{16}$, and $\frac{1}{64}$.

LOOKING FOR STRUCTURE

When the terms of a geometric sequence alternate between positive and negative terms, or vice versa, the common ratio is negative.

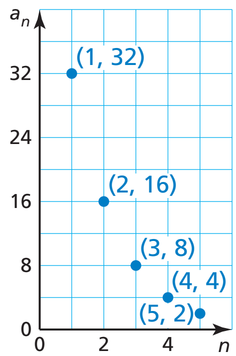
EXAMPLE 3 Graphing a Geometric Sequence

Graph the geometric sequence 32, 16, 8, 4, 2, ... What do you notice?

SOLUTION

Make a table. Then plot the ordered pairs (n, a_n).

Position, n	1	2	3	4	5
Term, a_n	32	16	8	4	2



▶ The points appear to lie on an exponential curve.

STUDY TIP

The points of any geometric sequence with a *positive* common ratio lie on an exponential curve.

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Write the next three terms of the geometric sequence. Then graph the sequence.

4. 1, 3, 9, 27, . . .

5. 2500, 500, 100, 20, . . .

6. 80, −40, 20, −10, . . .

7. −2, 4, −8, 16, . . .

Writing Geometric Sequences as Functions

Because consecutive terms of a geometric sequence have a common ratio, you can use the first term a_1 and the common ratio r to write an exponential function that describes a geometric sequence. Let $a_1 = 1$ and $r = 5$.

Position, n	Term, a_n	Written using a_1 and r	Numbers
1	first term, a_1	a_1	1
2	second term, a_2	$a_1 r$	$1 \cdot 5 = 5$
3	third term, a_3	$a_1 r^2$	$1 \cdot 5^2 = 25$
4	fourth term, a_4	$a_1 r^3$	$1 \cdot 5^3 = 125$
⋮	⋮	⋮	⋮
n	nth term, a_n	$a_1 r^{n-1}$	$1 \cdot 5^{n-1}$

STUDY TIP

Notice that the equation $a_n = a_1 r^{n-1}$ is of the form $y = ab^x$.

Core Concept

Equation for a Geometric Sequence

Let a_n be the nth term of a geometric sequence with first term a_1 and common ratio r. The nth term is given by

$$a_n = a_1 r^{n-1}.$$

EXAMPLE 4 Finding the *n*th Term of a Geometric Sequence

Write an equation for the nth term of the geometric sequence 2, 12, 72, 432, ... Then find a_{10}.

SOLUTION

The first term is 2, and the common ratio is 6.

$a_n = a_1 r^{n-1}$ Equation for a geometric sequence

$a_n = 2(6)^{n-1}$ Substitute 2 for a_1 and 6 for r.

Use the equation to find the 10th term.

$a_n = 2(6)^{n-1}$ Write the equation.

$a_{10} = 2(6)^{10-1}$ Substitute 10 for n.

$= 20{,}155{,}392$ Simplify.

The 10th term of the geometric sequence is 20,155,392.

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Write an equation for the nth term of the geometric sequence. Then find a_7.

8. 1, −5, 25, −125, . . .

9. 13, 26, 52, 104, . . .

10. 432, 72, 12, 2, . . .

11. 4, 10, 25, 62.5, . . .

You can rewrite the equation for a geometric sequence with first term a_1 and common ratio r in function notation by replacing a_n with $f(n)$.

$$f(n) = a_1 r^{n-1}$$

The domain of the function is the set of positive integers.

EXAMPLE 5 Modeling with Mathematics

Clicking the *zoom-out* button on a mapping website doubles the side length of the square map. After how many clicks on the *zoom-out* button is the side length of the map 640 miles?

Zoom-out clicks	1	2	3
Map side length (miles)	5	10	20

SOLUTION

1. **Understand the Problem** You know that the side length of the square map doubles after each click on the *zoom-out* button. So, the side lengths of the map represent the terms of a geometric sequence. You need to find the number of clicks it takes for the side length of the map to be 640 miles.

2. **Make a Plan** Begin by writing a function f for the nth term of the geometric sequence. Then find the value of n for which $f(n) = 640$.

3. **Solve the Problem** The first term is 5, and the common ratio is 2.

$f(n) = a_1 r^{n-1}$ Function for a geometric sequence

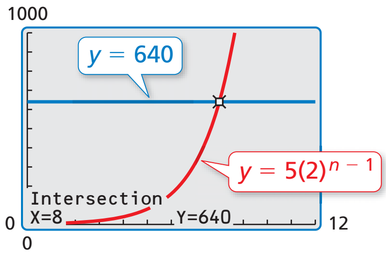
$f(n) = 5(2)^{n-1}$ Substitute 5 for a_1 and 2 for r.

The function $f(n) = 5(2)^{n-1}$ represents the geometric sequence. Use this function to find the value of n for which $f(n) = 640$. So, use the equation $640 = 5(2)^{n-1}$ to write a system of equations.

$y = 5(2)^{n-1}$ Equation 1

$y = 640$ Equation 2

Then use a graphing calculator to graph the equations and find the point of intersection. The point of intersection is (8, 640).



▶ So, after eight clicks, the side length of the map is 640 miles.

4. **Look Back** Find the value of n for which $f(n) = 640$ algebraically.

$640 = 5(2)^{n-1}$ Write the equation.

$128 = (2)^{n-1}$ Divide each side by 5.

$2^7 = (2)^{n-1}$ Rewrite 128 as 2^7.

$7 = n - 1$ Equate the exponents.

$8 = n$ ✓ Add 1 to each side.

USING APPROPRIATE TOOLS STRATEGICALLY

You can also use the *table* feature of a graphing calculator to find the value of n for which $f(n) = 640$.

X	Y1	Y2
3	20	640
4	40	640
5	80	640
6	160	640
7	320	640
8	640	640
9	1280	640

X=8

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12. **WHAT IF?** After how many clicks on the *zoom-out* button is the side length of the map 2560 miles?

6.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Compare the two sequences.

   2, 4, 6, 8, 10, . . .   2, 4, 8, 16, 32, . . .

2. **CRITICAL THINKING** Why do the points of a geometric sequence lie on an exponential curve only when the common ratio is positive?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the common ratio of the geometric sequence.

3. 4, 12, 36, 108, . . .

4. 36, 6, 1, $\frac{1}{6}$, . . .

5. $\frac{3}{8}$, -3, 24, -192, . . .

6. 0.1, 1, 10, 100, . . .

7. 128, 96, 72, 54, . . .

8. -162, 54, -18, 6, . . .

In Exercises 9–14, determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. *(See Example 1.)*

9. -8, 0, 8, 16, . . .

10. -1, 4, -7, 10, . . .

11. 9, 14, 20, 27, . . .

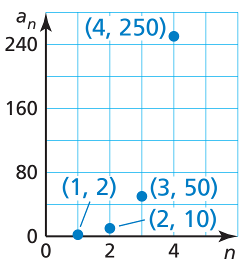
12. $\frac{3}{49}$, $\frac{3}{7}$, 3, 21, . . .

13. 192, 24, 3, $\frac{3}{8}$, . . .

14. -25, -18, -11, -4, . . .

In Exercises 15–18, determine whether the graph represents an *arithmetic sequence*, a *geometric sequence*, or *neither*. Explain your reasoning.

15.



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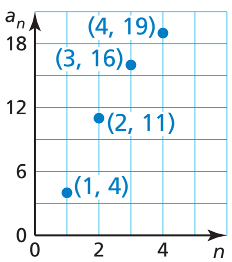

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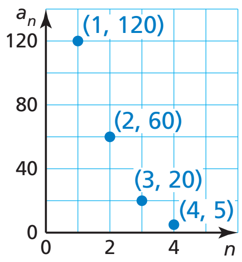

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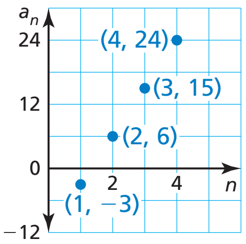


In Exercises 19–24, write the next three terms of the geometric sequence. Then graph the sequence. *(See Examples 2 and 3.)*

19. 5, 20, 80, 320, . . .

20. -3, 12, -48, 192, . . .

21. 81, -27, 9, -3, . . .

22. -375, -75, -15, -3, . . .

23. 32, 8, 2, $\frac{1}{2}$, . . .

24. $\frac{16}{9}$, $\frac{8}{3}$, 4, 6, . . .

In Exercises 25–32, write an equation for the *n*th term of the geometric sequence. Then find a_6. *(See Example 4.)*

25. 2, 8, 32, 128, . . .

26. 0.6, -3, 15, -75, . . .

27. $-\frac{1}{8}$, $-\frac{1}{4}$, $-\frac{1}{2}$, -1, . . .

28. 0.1, 0.9, 8.1, 72.9, . . .

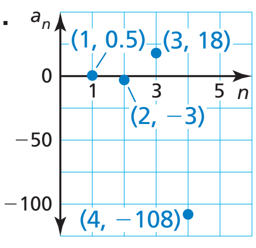
29.

n	1	2	3	4
a_n	7640	764	76.4	7.64

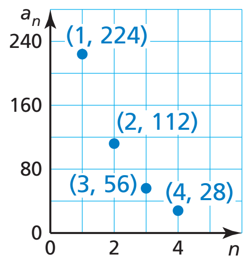
30.

n	1	2	3	4
a_n	-192	48	-12	3

31.

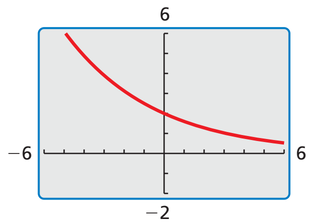


32.



33. **PROBLEM SOLVING** A badminton tournament begins with 128 teams. After the first round, 64 teams remain. After the second round, 32 teams remain. How many teams remain after the third, fourth, and fifth rounds?

34. PROBLEM SOLVING The graphing calculator screen displays an area of 96 square units. After you zoom out once, the area is 384 square units. After you zoom out a second time, the area is 1536 square units. What is the screen area after you zoom out four times?



35. ERROR ANALYSIS Describe and correct the error in writing the next three terms of the geometric sequence.

$$-8, \underset{\times(-2)}{\curvearrowright} 4, \underset{\times(-2)}{\curvearrowright} -2, \underset{\times(-2)}{\curvearrowright} 1, \ldots$$

The next three terms are -2, 4, and -8.

36. ERROR ANALYSIS Describe and correct the error in writing an equation for the nth term of the geometric sequence.

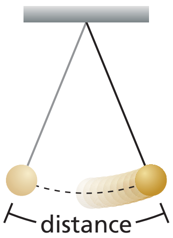
$-2, -12, -72, -432, \ldots$

The first term is -2, and the common ratio is -6.

$$a_n = a_1 r^{n-1}$$
$$a_n = -2(-6)^{n-1}$$

37. MODELING WITH MATHEMATICS The distance (in millimeters) traveled by a swinging pendulum decreases after each swing, as shown in the table. *(See Example 5.)*

Swing	1	2	3
Distance (in millimeters)	625	500	400



a. Write a function that represents the distance the pendulum swings on its nth swing.

b. After how many swings is the distance 256 millimeters?

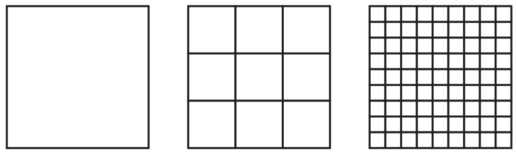
38. MODELING WITH MATHEMATICS You start a chain email and send it to six friends. The next day, each of your friends forwards the email to six people. The process continues for a few days.

a. Write a function that represents the number of people who have received the email after n days.

b. After how many days will 1296 people have received the email?

MATHEMATICAL CONNECTIONS **In Exercises 39 and 40, (a) write a function that represents the sequence of figures and (b) describe the 10th figure in the sequence.**

39.



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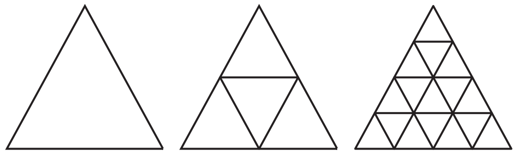


41. REASONING Write a sequence that represents the number of teams that have been eliminated after n rounds of the badminton tournament in Exercise 33. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

42. REASONING Write a sequence that represents the perimeter of the graphing calculator screen in Exercise 34 after you zoom out n times. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

43. WRITING Compare the graphs of arithmetic sequences to the graphs of geometric sequences.

44. MAKING AN ARGUMENT You are given two consecutive terms of a sequence.

$$\ldots, -8, 0, \ldots$$

Your friend says that the sequence is not geometric. A classmate says that is impossible to know given only two terms. Who is correct? Explain.

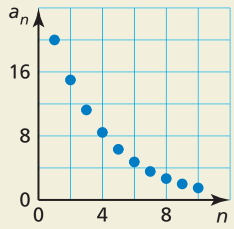
45. **CRITICAL THINKING** Is the sequence shown an *arithmetic* sequence? a *geometric* sequence? Explain your reasoning.

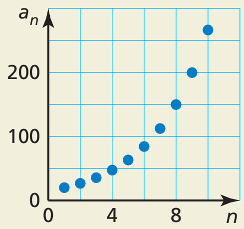
$3, 3, 3, 3, \ldots$

46. **HOW DO YOU SEE IT?** Without performing any calculations, match each equation with its graph. Explain your reasoning.

a. $a_n = 20\left(\frac{4}{3}\right)^{n-1}$

b. $a_n = 20\left(\frac{3}{4}\right)^{n-1}$

A.


B.


47. **REASONING** What is the 9th term of the geometric sequence where $a_3 = 81$ and $r = 3$?

48. **OPEN-ENDED** Write a sequence that has a pattern but is not arithmetic or geometric. Describe the pattern.

49. **ATTENDING TO PRECISION** Are the terms of a geometric sequence independent or dependent? Explain your reasoning.

50. **DRAWING CONCLUSIONS** A college student makes a deal with her parents to live at home instead of living on campus. She will pay her parents \$0.01 for the first day of the month, \$0.02 for the second day, \$0.04 for the third day, and so on.

a. Write an equation that represents the *n*th term of the geometric sequence.

b. What will she pay on the 25th day?

c. Did the student make a good choice or should she have chosen to live on campus? Explain.

51. **REPEATED REASONING** A soup kitchen makes 16 gallons of soup. Each day, a quarter of the soup is served and the rest is saved for the next day.

a. Write the first five terms of the sequence of the number of fluid ounces of soup left each day.

b. Write an equation that represents the *n*th term of the sequence.

c. When is all the soup gone? Explain.

52. **THOUGHT PROVOKING** Find the sum of the terms of the geometric sequence.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}, \ldots$$

Explain your reasoning. Write a different infinite geometric sequence that has the same sum.

53. **OPEN-ENDED** Write a geometric sequence in which $a_2 < a_1 < a_3$.

54. **NUMBER SENSE** Write an equation that represents the *n*th term of each geometric sequence shown.

n	1	2	3	4
a_n	2	6	18	54

n	1	2	3	4
b_n	1	5	25	125

a. Do the terms $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

b. Do the terms $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use residuals to determine whether the model is a good fit for the data in the table. Explain. *(Section 4.5)*

55. $y = 3x - 8$

x	0	1	2	3	4	5	6
y	−10	−2	−1	2	1	7	10

56. $y = -5x + 1$

x	−3	−2	−1	0	1	2	3
y	6	4	6	1	2	−4	−3