4 Transformations

Magnification *(p. 213)*

Photo Stickers *(p. 211)*

Kaleidoscope *(p. 196)*

Revolving Door *(p. 195)*

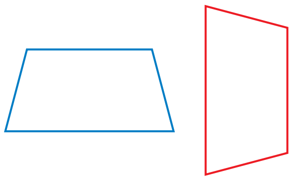
Chess *(p. 179)*

Maintaining Mathematical Proficiency

Identifying Transformations

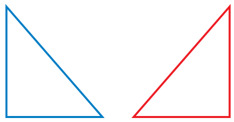
Example 1 **Tell whether the red figure is a translation, reflection, rotation, or dilation of the blue figure.**

a.



The blue figure turns to form the red figure, so it is a rotation.

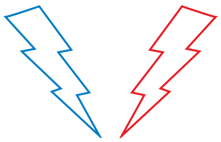
b.



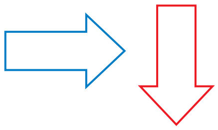
The red figure is a mirror image of the blue figure, so it is a reflection.

Tell whether the red figure is a translation, reflection, rotation, or dilation of the blue figure.

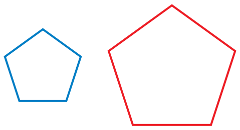
1.



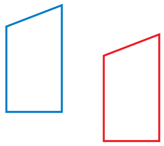
2.



3.

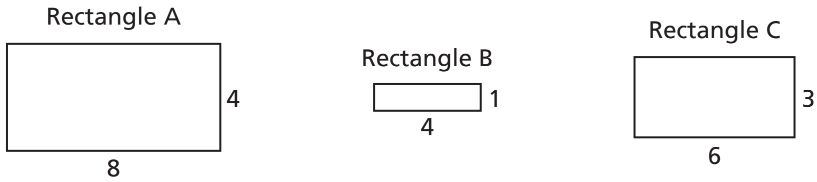


4.



Identifying Similar Figures

Example 2 **Which rectangle is similar to Rectangle A?**



Each figure is a rectangle, so corresponding angles are congruent. Check to see whether corresponding side lengths are proportional.

Rectangle A and Rectangle B

$$\frac{\text{Length of A}}{\text{Length of B}} = \frac{8}{4} = 2 \qquad \frac{\text{Width of A}}{\text{Width of B}} = \frac{4}{1} = 4$$

not proportional

Rectangle A and Rectangle C

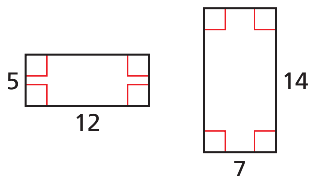
$$\frac{\text{Length of A}}{\text{Length of C}} = \frac{8}{6} = \frac{4}{3} \qquad \frac{\text{Width of A}}{\text{Width of C}} = \frac{4}{3}$$

proportional

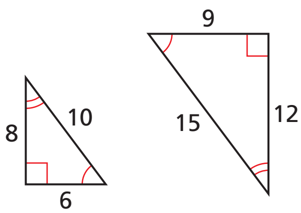
▶ So, Rectangle C is similar to Rectangle A.

Tell whether the two figures are similar. Explain your reasoning.

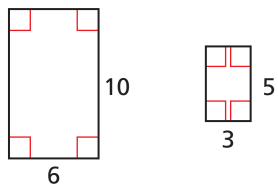
5.



6.



7.



8. **ABSTRACT REASONING** Can you draw two squares that are not similar? Explain your reasoning.

Mathematical Practices

Mathematically proficient students use dynamic geometry software strategically.

Using Dynamic Geometry Software

Core Concept

Using Dynamic Geometry Software

Dynamic geometry software allows you to create geometric drawings, including:

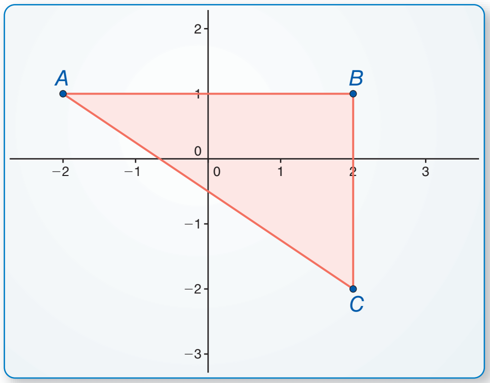
- drawing a point
- drawing a line
- drawing a line segment
- drawing an angle
- measuring an angle
- measuring a line segment
- drawing a circle
- drawing an ellipse
- drawing a perpendicular line
- drawing a polygon
- copying and sliding an object
- reflecting an object in a line

EXAMPLE 1 Finding Side Lengths and Angle Measures

Use dynamic geometry software to draw a triangle with vertices at $A(-2, 1)$, $B(2, 1)$, and $C(2, -2)$. Find the side lengths and angle measures of the triangle.

SOLUTION

Using dynamic geometry software, you can create $\triangle ABC$, as shown.



Sample

Points
$A(-2, 1)$
$B(2, 1)$
$C(2, -2)$
Segments
$AB = 4$
$BC = 3$
$AC = 5$
Angles
$m\angle A = 36.87°$
$m\angle B = 90°$
$m\angle C = 53.13°$

▶ From the display, the side lengths are $AB = 4$ units, $BC = 3$ units, and $AC = 5$ units. The angle measures, rounded to two decimal places, are $m\angle A \approx 36.87°$, $m\angle B = 90°$, and $m\angle C \approx 53.13°$.

Monitoring Progress

Use dynamic geometry software to draw the polygon with the given vertices. Use the software to find the side lengths and angle measures of the polygon. Round your answers to the nearest hundredth.

1. $A(0, 2)$, $B(3, -1)$, $C(4, 3)$
2. $A(-2, 1)$, $B(-2, -1)$, $C(3, 2)$
3. $A(1, 1)$, $B(-3, 1)$, $C(-3, -2)$, $D(1, -2)$
4. $A(1, 1)$, $B(-3, 1)$, $C(-2, -2)$, $D(2, -2)$
5. $A(-3, 0)$, $B(0, 3)$, $C(3, 0)$, $D(0, -3)$
6. $A(0, 0)$, $B(4, 0)$, $C(1, 1)$, $D(0, 3)$

4.1 Translations

Essential Question How can you translate a figure in a coordinate plane?

EXPLORATION 1 Translating a Triangle in a Coordinate Plane

Work with a partner.

a. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

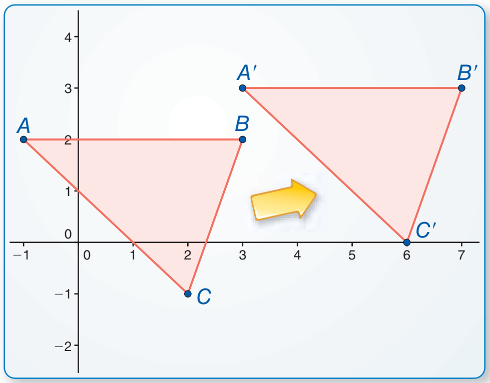
b. Copy the triangle and *translate* (or slide) it to form a new figure, called an *image*, $\triangle A'B'C'$ (read as "triangle A prime, B prime, C prime").

c. What is the relationship between the coordinates of the vertices of $\triangle ABC$ and those of $\triangle A'B'C'$?

d. What do you observe about the side lengths and angle measures of the two triangles?

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use appropriate tools strategically, including dynamic geometry software.

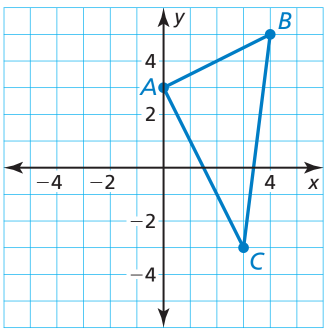


Sample

Points
$A(-1, 2)$
$B(3, 2)$
$C(2, -1)$
Segments
$AB = 4$
$BC = 3.16$
$AC = 4.24$
Angles
$m\angle A = 45°$
$m\angle B = 71.57°$
$m\angle C = 63.43°$

EXPLORATION 2 Translating a Triangle in a Coordinate Plane

Work with a partner.



a. The point (x, y) is translated a units horizontally and b units vertically. Write a rule to determine the coordinates of the image of (x, y).

$$(x, y) \rightarrow (\square, \square)$$

b. Use the rule you wrote in part (a) to translate $\triangle ABC$ 4 units left and 3 units down. What are the coordinates of the vertices of the image, $\triangle A'B'C'$?

c. Draw $\triangle A'B'C'$. Are its side lengths the same as those of $\triangle ABC$? Justify your answer.

EXPLORATION 3 Comparing Angles of Translations

Work with a partner.

a. In Exploration 2, is $\triangle ABC$ a right triangle? Justify your answer.

b. In Exploration 2, is $\triangle A'B'C'$ a right triangle? Justify your answer.

c. Do you think translations always preserve angle measures? Explain your reasoning.

Communicate Your Answer

4. How can you translate a figure in a coordinate plane?

5. In Exploration 2, translate $\triangle A'B'C'$ 3 units right and 4 units up. What are the coordinates of the vertices of the image, $\triangle A''B''C''$? How are these coordinates related to the coordinates of the vertices of the original triangle, $\triangle ABC$?

4.1 Lesson

Core Vocabulary

vector, *p. 174*
initial point, *p. 174*
terminal point, *p. 174*
horizontal component, *p. 174*
vertical component, *p. 174*
component form, *p. 174*
transformation, *p. 174*
image, *p. 174*
preimage, *p. 174*
translation, *p. 174*
rigid motion, *p. 176*
composition of transformations, *p. 176*

What You Will Learn

- Perform translations.
- Perform compositions.
- Solve real-life problems involving compositions.

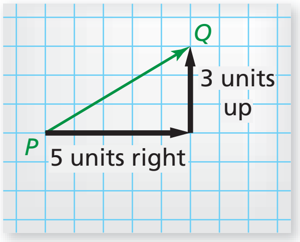
Performing Translations

A **vector** is a quantity that has both direction and *magnitude*, or size, and is represented in the coordinate plane by an arrow drawn from one point to another.

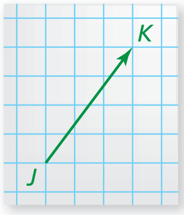
Core Concept

Vectors

The diagram shows a vector. The **initial point**, or starting point, of the vector is P, and the **terminal point**, or ending point, is Q. The vector is named $\overrightarrow{PQ}$, which is read as "vector PQ." The **horizontal component** of $\overrightarrow{PQ}$ is 5, and the **vertical component** is 3. The **component form** of a vector combines the horizontal and vertical components. So, the component form of $\overrightarrow{PQ}$ is $\langle 5, 3\rangle$.



EXAMPLE 1 Identifying Vector Components



In the diagram, name the vector and write its component form.

SOLUTION

The vector is $\overrightarrow{JK}$. To move from the initial point J to the terminal point K, you move 3 units right and 4 units up. So, the component form is $\langle 3, 4\rangle$.

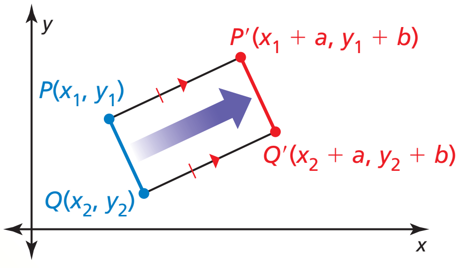
A **transformation** is a function that moves or changes a figure in some way to produce a new figure called an **image**. Another name for the original figure is the **preimage**. The points on the preimage are the inputs for the transformation, and the points on the image are the outputs.

Core Concept

Translations

A **translation** moves every point of a figure the same distance in the same direction. More specifically, a translation *maps*, or moves, the points P and Q of a plane figure along a vector $\langle a, b\rangle$ to the points P' and Q', so that one of the following statements is true.

- $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$, or
- $PP' = QQ'$ and $\overline{PP'}$ and $\overline{QQ'}$ are collinear.



STUDY TIP

You can use *prime notation* to name an image. For example, if the preimage is point P, then its image is point P', read as "point P *prime*."

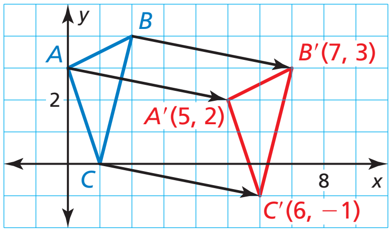
Translations map lines to parallel lines and segments to parallel segments. For instance, in the figure above, $\overline{PQ} \parallel \overline{P'Q'}$.

EXAMPLE 2 Translating a Figure Using a Vector

The vertices of $\triangle ABC$ are $A(0, 3)$, $B(2, 4)$, and C(1, 0). Translate $\triangle ABC$ using the vector $\langle 5, -1 \rangle$.

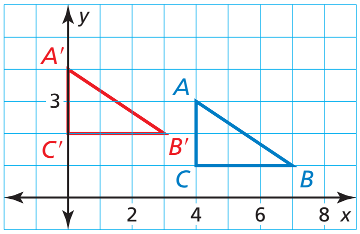
SOLUTION

First, graph $\triangle ABC$. Use $\langle 5, -1 \rangle$ to move each vertex 5 units right and 1 unit down. Label the image vertices. Draw $\triangle A'B'C'$. Notice that the vectors drawn from preimage vertices to image vertices are parallel.



You can also express a translation along the vector $\langle a, b \rangle$ using a rule, which has the notation $(x, y) \to (x + a, y + b)$.

EXAMPLE 3 Writing a Translation Rule



Write a rule for the translation of $\triangle ABC$ to $\triangle A'B'C'$.

SOLUTION

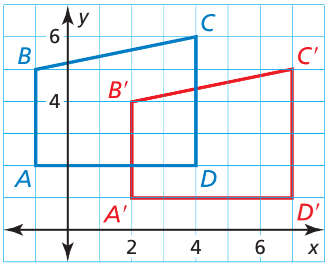
To go from A to A', you move 4 units left and 1 unit up, so you move along the vector $\langle -4, 1 \rangle$.

▶ So, a rule for the translation is $(x, y) \to (x - 4, y + 1)$.

EXAMPLE 4 Translating a Figure in the Coordinate Plane

Graph quadrilateral $ABCD$ with vertices $A(-1, 2)$, $B(-1, 5)$, $C(4, 6)$, and $D(4, 2)$ and its image after the translation $(x, y) \to (x + 3, y - 1)$.

SOLUTION



Graph quadrilateral $ABCD$. To find the coordinates of the vertices of the image, add 3 to the x-coordinates and subtract 1 from the y-coordinates of the vertices of the preimage. Then graph the image, as shown at the left.

$$(x, y) \to (x + 3, y - 1)$$

$$A(-1, 2) \to A'(2, 1)$$

$$B(-1, 5) \to B'(2, 4)$$

$$C(4, 6) \to C'(7, 5)$$

$$D(4, 2) \to D'(7, 1)$$

Monitoring Progress

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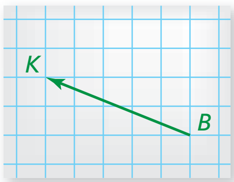


1. Name the vector and write its component form.
2. The vertices of $\triangle LMN$ are $L(2, 2)$, $M(5, 3)$, and $N(9, 1)$. Translate $\triangle LMN$ using the vector $\langle -2, 6 \rangle$.
3. In Example 3, write a rule to translate $\triangle A'B'C'$ back to $\triangle ABC$.
4. Graph $\triangle RST$ with vertices $R(2, 2)$, $S(5, 2)$, and $T(3, 5)$ and its image after the translation $(x, y) \to (x + 1, y + 2)$.

Performing Compositions

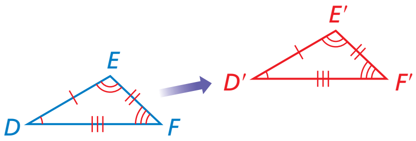
A **rigid motion** is a transformation that preserves length and angle measure. Another name for a rigid motion is an *isometry*. A rigid motion maps lines to lines, rays to rays, and segments to segments.

Postulate

Postulate 4.1 Translation Postulate

A translation is a rigid motion.

Because a translation is a rigid motion, and a rigid motion preserves length and angle measure, the following statements are true for the translation shown.



- $DE = D'E'$, $EF = E'F'$, $FD = F'D'$
- $m\angle D = m\angle D'$, $m\angle E = m\angle E'$, $m\angle F = m\angle F'$

When two or more transformations are combined to form a single transformation, the result is a **composition of transformations**.

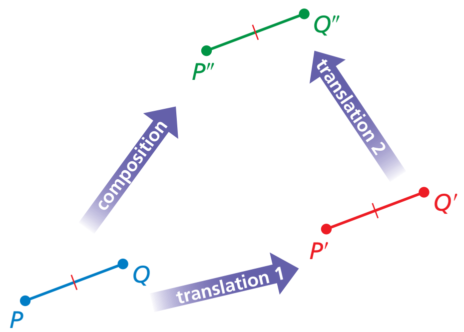
Theorem

Theorem 4.1 Composition Theorem

The composition of two (or more) rigid motions is a rigid motion.

Proof Ex. 35, p. 180

The theorem above is important because it states that no matter how many rigid motions you perform, lengths and angle measures will be preserved in the final image. For instance, the composition of two or more translations is a translation, as shown.



EXAMPLE 5 Performing a Composition

Graph $\overline{RS}$ with endpoints $R(-8, 5)$ and $S(-6, 8)$ and its image after the composition.

Translation: $(x, y) \to (x + 5, y - 2)$

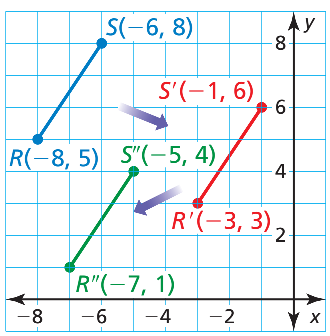
Translation: $(x, y) \to (x - 4, y - 2)$

SOLUTION

Step 1 Graph $\overline{RS}$.

Step 2 Translate $\overline{RS}$ 5 units right and 2 units down. $\overline{R'S'}$ has endpoints $R'(-3, 3)$ and $S'(-1, 6)$.

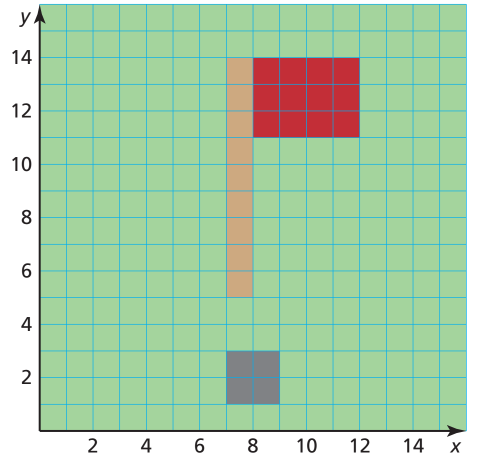
Step 3 Translate $\overline{R'S'}$ 4 units left and 2 units down. $\overline{R''S''}$ has endpoints $R''(-7, 1)$ and $S''(-5, 4)$.



Solving Real-Life Problems

EXAMPLE 6 Modeling with Mathematics

You are designing a favicon for a golf website. In an image-editing program, you move the red rectangle 2 units left and 3 units down. Then you move the red rectangle 1 unit right and 1 unit up. Rewrite the composition as a single translation.



SOLUTION

1. **Understand the Problem** You are given two translations. You need to rewrite the result of the composition of the two translations as a single translation.

2. **Make a Plan** You can choose an arbitrary point (x, y) in the red rectangle and determine the horizontal and vertical shift in the coordinates of the point after both translations. This tells you how much you need to shift each coordinate to map the original figure to the final image.

3. **Solve the Problem** Let $A(x, y)$ be an arbitrary point in the red rectangle. After the first translation, the coordinates of its image are

   $A'(x - 2, y - 3)$.

   The second translation maps $A'(x - 2, y - 3)$ to

   $A''(x - 2 + 1, y - 3 + 1) = A''(x - 1, y - 2)$.

   The composition of translations uses the original point (x, y) as the input and returns the point $(x - 1, y - 2)$ as the output.

   ▶ So, the single translation rule for the composition is $(x, y) \rightarrow (x - 1, y - 2)$.

4. **Look Back** Check that the rule is correct by testing a point. For instance, $(10, 12)$ is a point in the red rectangle. Apply the two translations to $(10, 12)$.

   $(10, 12) \rightarrow (8, 9) \rightarrow (9, 10)$

   Does the final result match the rule you found in Step 3?

   $(10, 12) \rightarrow (10 - 1, 12 - 2) = (9, 10)$ ✓

Monitoring Progress

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5. Graph $\overline{TU}$ with endpoints $T(1, 2)$ and $U(4, 6)$ and its image after the composition.

   Translation: $(x, y) \rightarrow (x - 2, y - 3)$
   Translation: $(x, y) \rightarrow (x - 4, y + 5)$

6. Graph $\overline{VW}$ with endpoints $V(-6, -4)$ and $W(-3, 1)$ and its image after the composition.

   Translation: $(x, y) \rightarrow (x + 3, y + 1)$
   Translation: $(x, y) \rightarrow (x - 6, y - 4)$

7. In Example 6, you move the gray square 2 units right and 3 units up. Then you move the gray square 1 unit left and 1 unit down. Rewrite the composition as a single transformation.

4.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

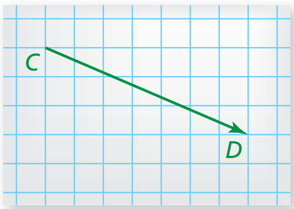
Vocabulary and Core Concept Check

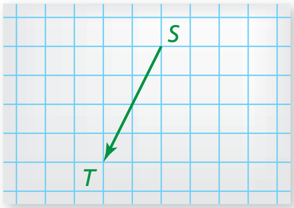
1. **VOCABULARY** Name the preimage and image of the transformation $\triangle ABC \rightarrow \triangle A'B'C'$.

2. **COMPLETE THE SENTENCE** A _______ moves every point of a figure the same distance in the same direction.

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, name the vector and write its component form. *(See Example 1.)*

3. 

4. 

In Exercises 5–8, the vertices of $\triangle DEF$ are $D(2, 5)$, $E(6, 3)$, and $F(4, 0)$. Translate $\triangle DEF$ using the given vector. Graph $\triangle DEF$ and its image. *(See Example 2.)*

5. $\langle 6, 0 \rangle$

6. $\langle 5, -1 \rangle$

7. $\langle -3, -7 \rangle$

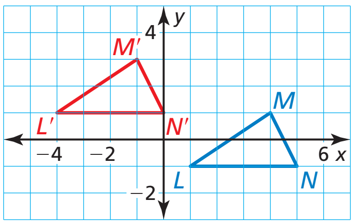
8. $\langle -2, -4 \rangle$

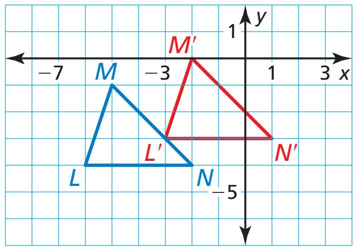
In Exercises 9 and 10, find the component form of the vector that translates $P(-3, 6)$ to P'.

9. $P'(0, 1)$

10. $P'(-4, 8)$

In Exercises 11 and 12, write a rule for the translation of $\triangle LMN$ to $\triangle L'M'N'$. *(See Example 3.)*

11. 

12. 

In Exercises 13–16, use the translation.

$$(x, y) \rightarrow (x - 8, y + 4)$$

13. What is the image of $A(2, 6)$?

14. What is the image of $B(-1, 5)$?

15. What is the preimage of $C'(-3, -10)$?

16. What is the preimage of $D'(4, -3)$?

In Exercises 17–20, graph $\triangle PQR$ with vertices $P(-2, 3)$, $Q(1, 2)$, and $R(3, -1)$ and its image after the translation. *(See Example 4.)*

17. $(x, y) \rightarrow (x + 4, y + 6)$

18. $(x, y) \rightarrow (x + 9, y - 2)$

19. $(x, y) \rightarrow (x - 2, y - 5)$

20. $(x, y) \rightarrow (x - 1, y + 3)$

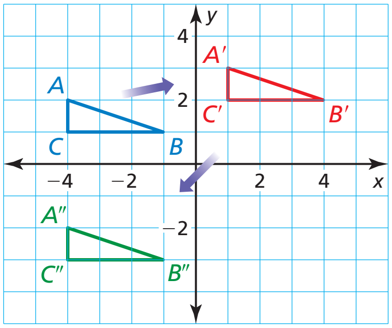
In Exercises 21 and 22, graph $\triangle XYZ$ with vertices $X(2, 4)$, $Y(6, 0)$, and $Z(7, 2)$ and its image after the composition. *(See Example 5.)*

21. **Translation:** $(x, y) \rightarrow (x + 12, y + 4)$
Translation: $(x, y) \rightarrow (x - 5, y - 9)$

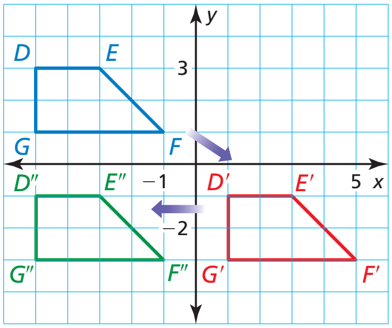
22. **Translation:** $(x, y) \rightarrow (x - 6, y)$
Translation: $(x, y) \rightarrow (x + 2, y + 7)$

In Exercises 23 and 24, describe the composition of translations.

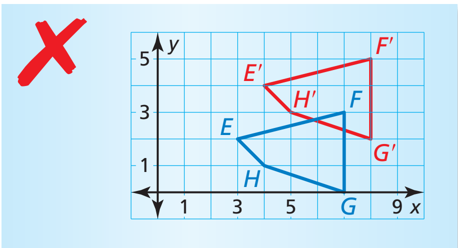
23.



24.

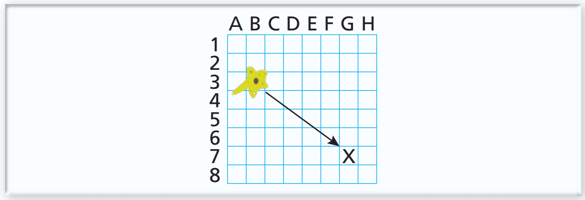


25. ERROR ANALYSIS Describe and correct the error in graphing the image of quadrilateral *EFGH* after the translation $(x, y) \rightarrow (x - 1, y - 2)$.



26. MODELING WITH MATHEMATICS In chess, the knight (the piece shaped like a horse) moves in an L pattern. The board shows two consecutive moves of a black knight during a game. Write a composition of translations for the moves. Then rewrite the composition as a single translation that moves the knight from its original position to its ending position. *(See Example 6.)*

27. PROBLEM SOLVING You are studying an amoeba through a microscope. Suppose the amoeba moves on a grid-indexed microscope slide in a straight line from square B3 to square G7.



a. Describe the translation.

b. The side length of each grid square is 2 millimeters. How far does the amoeba travel?

c. The amoeba moves from square B3 to square G7 in 24.5 seconds. What is its speed in millimeters per second?

28. MATHEMATICAL CONNECTIONS Translation A maps (x, y) to $(x + n, y + t)$. Translation B maps (x, y) to $(x + s, y + m)$.

a. Translate a point using Translation A, followed by Translation B. Write an algebraic rule for the final image of the point after this composition.

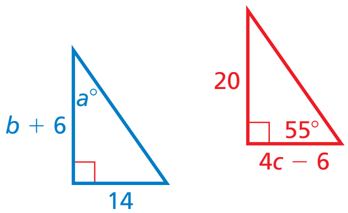
b. Translate a point using Translation B, followed by Translation A. Write an algebraic rule for the final image of the point after this composition.

c. Compare the rules you wrote for parts (a) and (b). Does it matter which translation you do first? Explain your reasoning.

MATHEMATICAL CONNECTIONS In Exercises 29 and 30, a translation maps the blue figure to the red figure. Find the value of each variable.

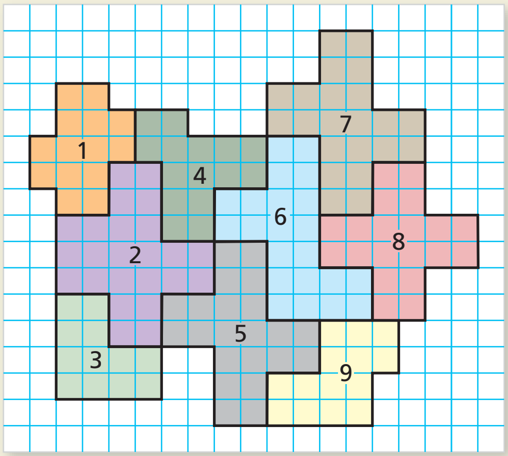
29.

30.



31. **USING STRUCTURE** Quadrilateral $DEFG$ has vertices $D(-1, 2)$, $E(-2, 0)$, $F(-1, -1)$, and $G(1, 3)$. A translation maps quadrilateral $DEFG$ to quadrilateral $D'E'F'G'$. The image of D is $D'(-2, -2)$. What are the coordinates of E', F', and G'?

32. **HOW DO YOU SEE IT?** Which two figures represent a translation? Describe the translation.



33. **REASONING** The translation $(x, y) \rightarrow (x + m, y + n)$ maps $\overline{PQ}$ to $\overline{P'Q'}$. Write a rule for the translation of $\overline{P'Q'}$ to $\overline{PQ}$. Explain your reasoning.

34. **DRAWING CONCLUSIONS** The vertices of a rectangle are $Q(2, -3)$, $R(2, 4)$, $S(5, 4)$, and $T(5, -3)$.

a. Translate rectangle $QRST$ 3 units left and 3 units down to produce rectangle $Q'R'S'T'$. Find the area of rectangle $QRST$ and the area of rectangle $Q'R'S'T'$.

b. Compare the areas. Make a conjecture about the areas of a preimage and its image after a translation.

35. **PROVING A THEOREM** Prove the Composition Theorem (Theorem 4.1).

36. **PROVING A THEOREM** Use properties of translations to prove each theorem.

a. Corresponding Angles Theorem (Theorem 3.1)

b. Corresponding Angles Converse (Theorem 3.5)

37. **WRITING** Explain how to use translations to draw a rectangular prism.

38. **MATHEMATICAL CONNECTIONS** The vector $\overrightarrow{PQ} = \langle 4, 1 \rangle$ describes the translation of $A(-1, w)$ onto $A'(2x + 1, 4)$ and $B(8y - 1, 1)$ onto $B'(3, 3z)$. Find the values of w, x, y, and z.

39. **MAKING AN ARGUMENT** A translation maps $\overline{GH}$ to $\overline{G'H'}$. Your friend claims that if you draw segments connecting G to G' and H to H', then the resulting quadrilateral is a parallelogram. Is your friend correct? Explain your reasoning.

40. **THOUGHT PROVOKING** You are a graphic designer for a company that manufactures floor tiles. Design a floor tile in a coordinate plane. Then use translations to show how the tiles cover an entire floor. Describe the translations that map the original tile to four other tiles.

41. **REASONING** The vertices of $\triangle ABC$ are $A(2, 2)$, $B(4, 2)$, and $C(3, 4)$. Graph the image of $\triangle ABC$ after the transformation $(x, y) \rightarrow (x + y, y)$. Is this transformation a translation? Explain your reasoning.

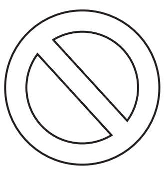
42. **PROOF** $\overline{MN}$ is perpendicular to line ℓ. $\overline{M'N'}$ is the translation of $\overline{MN}$ 2 units to the left. Prove that $\overline{M'N'}$ is perpendicular to ℓ.

Maintaining Mathematical Proficiency

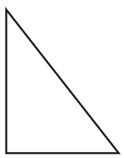
Reviewing what you learned in previous grades and lessons

Tell whether the figure can be folded in half so that one side matches the other. *(Skills Review Handbook)*

43.



44.



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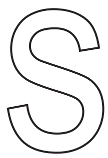
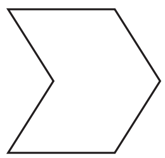


46.



Simplify the expression. *(Skills Review Handbook)*

47. $-(-x)$

48. $-(x + 3)$

49. $x - (12 - 5x)$

50. $x - (-2x + 4)$

4.2 Reflections

Essential Question How can you reflect a figure in a coordinate plane?

EXPLORATION 1 Reflecting a Triangle Using a Reflective Device

Work with a partner. Use a straightedge to draw any triangle on paper. Label it $\triangle ABC$.

a. Use the straightedge to draw a line that does not pass through the triangle. Label it m.

b. Place a reflective device on line m.

c. Use the reflective device to plot the images of the vertices of $\triangle ABC$. Label the images of vertices A, B, and C as A', B', and C', respectively.

d. Use a straightedge to draw $\triangle A'B'C'$ by connecting the vertices.

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Reflecting a Triangle in a Coordinate Plane

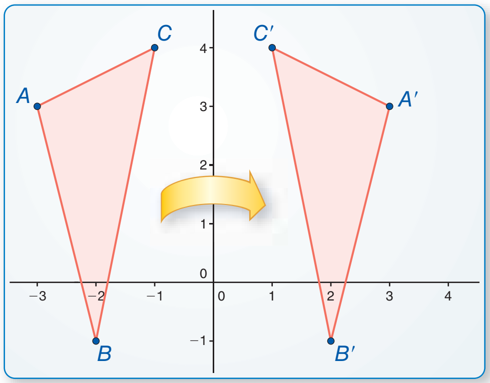
Work with a partner. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

a. *Reflect* $\triangle ABC$ in the y-axis to form $\triangle A'B'C'$.

b. What is the relationship between the coordinates of the vertices of $\triangle ABC$ and those of $\triangle A'B'C'$?

c. What do you observe about the side lengths and angle measures of the two triangles?

d. *Reflect* $\triangle ABC$ in the x-axis to form $\triangle A'B'C'$. Then repeat parts (b) and (c).



Sample

Points
$A(-3, 3)$
$B(-2, -1)$
$C(-1, 4)$
Segments
$AB = 4.12$
$BC = 5.10$
$AC = 2.24$
Angles
$m\angle A = 102.53°$
$m\angle B = 25.35°$
$m\angle C = 52.13°$

Communicate Your Answer

3. How can you reflect a figure in a coordinate plane?

4.2 Lesson

Core Vocabulary

reflection, *p. 182*
line of reflection, *p. 182*
glide reflection, *p. 184*
line symmetry, *p. 185*
line of symmetry, *p. 185*

What You Will Learn

- Perform reflections.
- Perform glide reflections.
- Identify lines of symmetry.
- Solve real-life problems involving reflections.

Performing Reflections

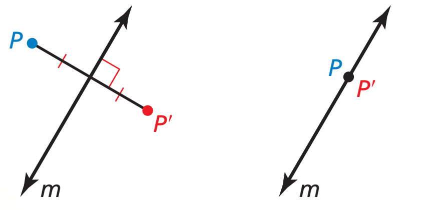
Core Concept

Reflections

A **reflection** is a transformation that uses a line like a mirror to reflect a figure. The mirror line is called the **line of reflection**.

A reflection in a line m maps every point P in the plane to a point P', so that for each point one of the following properties is true.

- If P is not on m, then m is the perpendicular bisector of $\overline{PP'}$, or
- If P is on m, then $P = P'$.



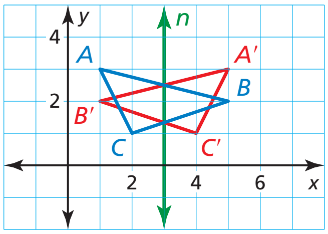
EXAMPLE 1 **Reflecting in Horizontal and Vertical Lines**

Graph $\triangle ABC$ with vertices $A(1, 3)$, $B(5, 2)$, and $C(2, 1)$ and its image after the reflection described.

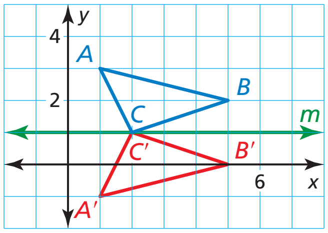
a. In the line n: $x = 3$ **b.** In the line m: $y = 1$

SOLUTION

a. Point A is 2 units left of line n, so its reflection A' is 2 units right of line n at $(5, 3)$. Also, B' is 2 units left of line n at $(1, 2)$, and C' is 1 unit right of line n at $(4, 1)$.



b. Point A is 2 units above line m, so A' is 2 units below line m at $(1, -1)$. Also, B' is 1 unit below line m at $(5, 0)$. Because point C is on line m, you know that $C = C'$.



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Graph $\triangle ABC$ from Example 1 and its image after a reflection in the given line.

1. $x = 4$

2. $x = -3$

3. $y = 2$

4. $y = -1$

EXAMPLE 2 Reflecting in the Line $y = x$

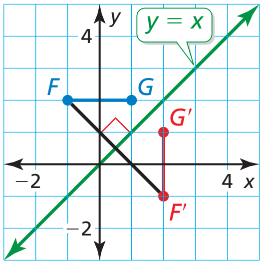
Graph $\overline{FG}$ with endpoints $F(-1, 2)$ and $G(1, 2)$ and its image after a reflection in the line $y = x$.

REMEMBER

The product of the slopes of perpendicular lines is -1.

SOLUTION

The slope of $y = x$ is 1. The segment from F to its image, $\overline{FF'}$, is perpendicular to the line of reflection $y = x$, so the slope of $\overline{FF'}$ will be -1 (because $1(-1) = -1$). From F, move 1.5 units right and 1.5 units down to $y = x$. From that point, move 1.5 units right and 1.5 units down to locate $F'(2, -1)$.



The slope of $\overline{GG'}$ will also be -1. From G, move 0.5 unit right and 0.5 unit down to $y = x$. Then move 0.5 unit right and 0.5 unit down to locate $G'(2, 1)$.

You can use coordinate rules to find the images of points reflected in four special lines.

Core Concept

Coordinate Rules for Reflections

- If (a, b) is reflected in the x-axis, then its image is the point $(a, -b)$.
- If (a, b) is reflected in the y-axis, then its image is the point $(-a, b)$.
- If (a, b) is reflected in the line $y = x$, then its image is the point (b, a).
- If (a, b) is reflected in the line $y = -x$, then its image is the point $(-b, -a)$.

EXAMPLE 3 Reflecting in the Line $y = -x$

Graph $\overline{FG}$ from Example 2 and its image after a reflection in the line $y = -x$.

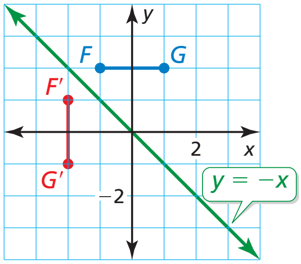
SOLUTION

Use the coordinate rule for reflecting in the line $y = -x$ to find the coordinates of the endpoints of the image. Then graph $\overline{FG}$ and its image.

$$(a, b) \rightarrow (-b, -a)$$

$$F(-1, 2) \rightarrow F'(-2, 1)$$

$$G(1, 2) \rightarrow G'(-2, -1)$$



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The vertices of $\triangle JKL$ are $J(1, 3)$, $K(4, 4)$, and $L(3, 1)$.

5. Graph $\triangle JKL$ and its image after a reflection in the x-axis.

6. Graph $\triangle JKL$ and its image after a reflection in the y-axis.

7. Graph $\triangle JKL$ and its image after a reflection in the line $y = x$.

8. Graph $\triangle JKL$ and its image after a reflection in the line $y = -x$.

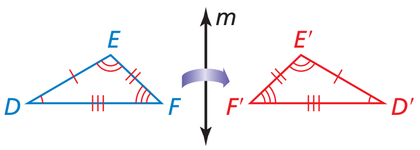
9. In Example 3, verify that $\overline{FF'}$ is perpendicular to $y = -x$.

Performing Glide Reflections

Postulate

Postulate 4.2 Reflection Postulate

A reflection is a rigid motion.



Because a reflection is a rigid motion, and a rigid motion preserves length and angle measure, the following statements are true for the reflection shown.

- $DE = D'E'$, $EF = E'F'$, $FD = F'D'$
- $m\angle D = m\angle D'$, $m\angle E = m\angle E'$, $m\angle F = m\angle F'$

Because a reflection is a rigid motion, the Composition Theorem (Theorem 4.1) guarantees that any composition of reflections and translations is a rigid motion.

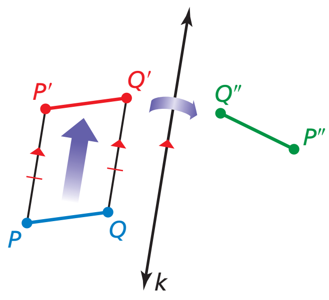
STUDY TIP

The line of reflection must be parallel to the direction of the translation to be a glide reflection.

A **glide reflection** is a transformation involving a translation followed by a reflection in which every point P is mapped to a point P'' by the following steps.

Step 1 First, a translation maps P to P'.

Step 2 Then, a reflection in a line k parallel to the direction of the translation maps P' to P''.



EXAMPLE 4 Performing a Glide Reflection

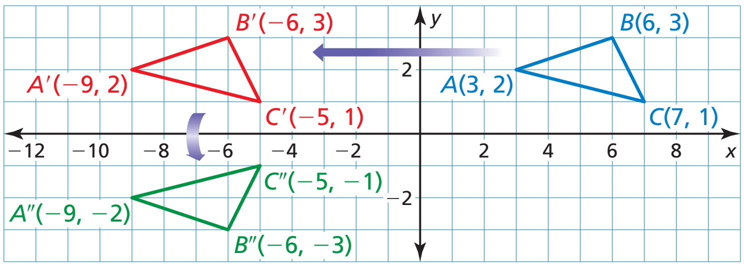
Graph $\triangle ABC$ with vertices $A(3, 2)$, $B(6, 3)$, and $C(7, 1)$ and its image after the glide reflection.

Translation: $(x, y) \rightarrow (x - 12, y)$

Reflection: in the x-axis

SOLUTION

Begin by graphing $\triangle ABC$. Then graph $\triangle A'B'C'$ after a translation 12 units left. Finally, graph $\triangle A''B''C''$ after a reflection in the x-axis.



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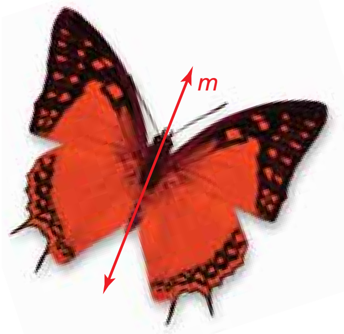
10. **WHAT IF?** In Example 4, $\triangle ABC$ is translated 4 units down and then reflected in the y-axis. Graph $\triangle ABC$ and its image after the glide reflection.

11. In Example 4, describe a glide reflection from $\triangle A''B''C''$ to $\triangle ABC$.

Identifying Lines of Symmetry

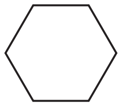


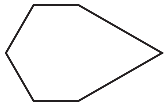
A figure in the plane has **line symmetry** when the figure can be mapped onto itself by a reflection in a line. This line of reflection is a **line of symmetry**, such as line m at the left. A figure can have more than one line of symmetry.

EXAMPLE 5 Identifying Lines of Symmetry

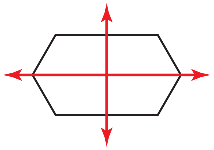
How many lines of symmetry does each hexagon have?

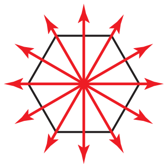
a. 

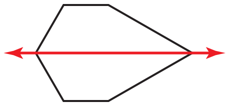
b. 

c. 

SOLUTION

a. 

b. 

c. 

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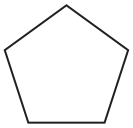
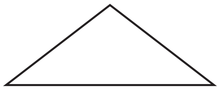
Determine the number of lines of symmetry for the figure.

12. 

13. 

14. 

15. Draw a hexagon with no lines of symmetry.

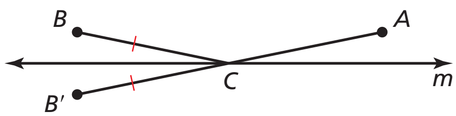
Solving Real-Life Problems

EXAMPLE 6 Finding a Minimum Distance

You are going to buy books. Your friend is going to buy CDs. Where should you park to minimize the distance you both will walk?

SOLUTION

Reflect B in line m to obtain B'. Then draw $\overline{AB'}$. Label the intersection of $\overline{AB'}$ and m as C. Because AB' is the shortest distance between A and B' and $BC = B'C$, park at point C to minimize the combined distance, $AC + BC$, you both have to walk.



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16. Look back at Example 6. Answer the question by using a reflection of point A instead of point B.

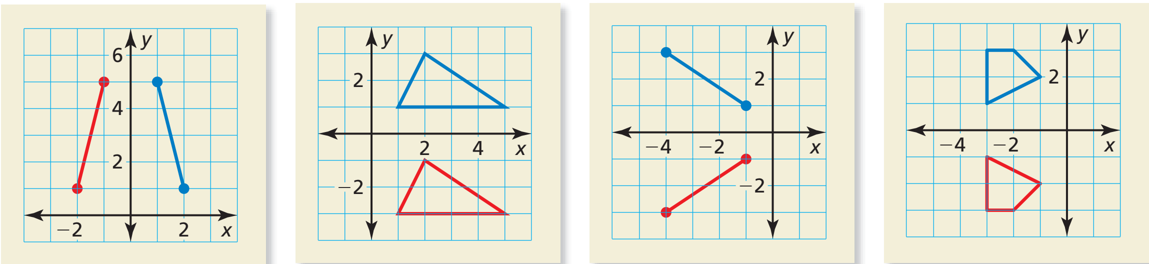
4.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** A glide reflection is a combination of which two transformations?

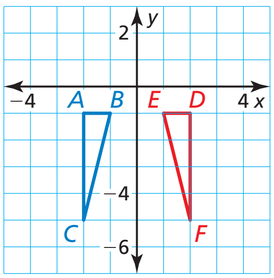
2. **WHICH ONE DOESN'T BELONG?** Which transformation does *not* belong with the other three? Explain your reasoning.



Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine whether the coordinate plane shows a reflection in the *x-axis*, *y-axis*, or *neither*.

3.



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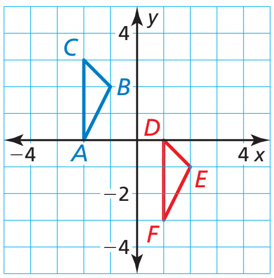

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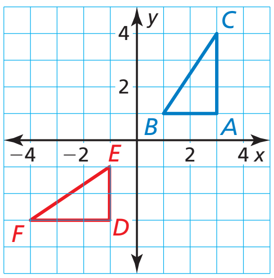

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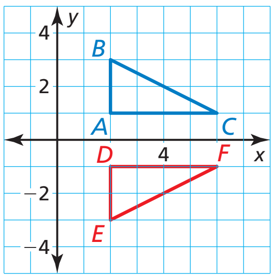


In Exercises 7–12, graph $\triangle JKL$ and its image after a reflection in the given line. *(See Example 1.)*

7. $J(2, -4)$, $K(3, 7)$, $L(6, -1)$; x-axis

8. $J(5, 3)$, $K(1, -2)$, $L(-3, 4)$; y-axis

9. $J(2, -1)$, $K(4, -5)$, $L(3, 1)$; $x = -1$

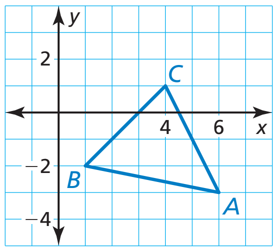
10. $J(1, -1)$, $K(3, 0)$, $L(0, -4)$; $x = 2$

11. $J(2, 4)$, $K(-4, -2)$, $L(-1, 0)$; $y = 1$

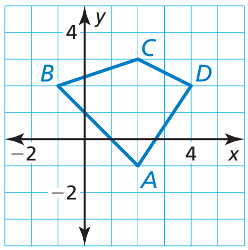
12. $J(3, -5)$, $K(4, -1)$, $L(0, -3)$; $y = -3$

In Exercises 13–16, graph the polygon and its image after a reflection in the given line. *(See Examples 2 and 3.)*

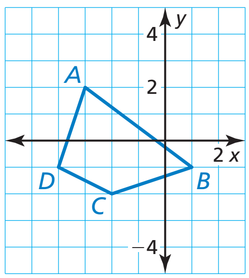
13. $y = x$



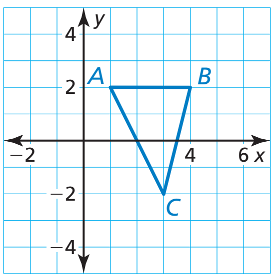
14. $y = x$



15. $y = -x$



16. $y = -x$



In Exercises 17–20, graph $\triangle RST$ with vertices $R(4, 1)$, $S(7, 3)$, and $T(6, 4)$ and its image after the glide reflection. *(See Example 4.)*

17. **Translation:** $(x, y) \rightarrow (x, y - 1)$
Reflection: in the y-axis

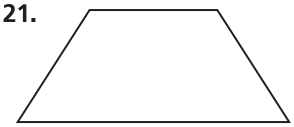
18. **Translation:** $(x, y) \rightarrow (x - 3, y)$
Reflection: in the line $y = -1$

19. **Translation:** $(x, y) \rightarrow (x, y + 4)$
Reflection: in the line $x = 3$

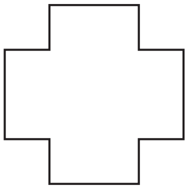
20. **Translation:** $(x, y) \rightarrow (x + 2, y + 2)$
Reflection: in the line $y = x$

In Exercises 21–24, determine the number of lines of symmetry for the figure. *(See Example 5.)*

21.

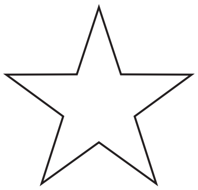


22.



23.

24.



25. **USING STRUCTURE** Identify the line symmetry (if any) of each word.

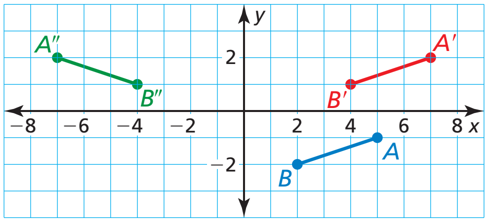
a. LOOK

b. MOM

c. OX

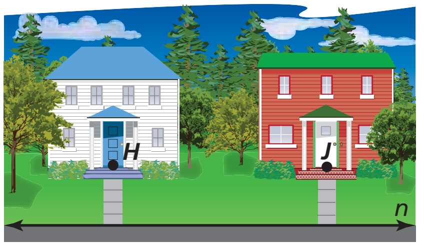
d. DAD

26. **ERROR ANALYSIS** Describe and correct the error in describing the transformation.

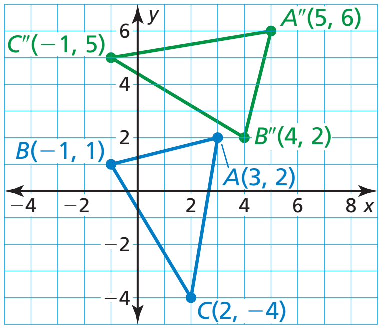


✗ $\overline{AB}$ to $\overline{A''B''}$ is a glide reflection.

27. **MODELING WITH MATHEMATICS** You park at some point K on line n. You deliver a pizza to House H, go back to your car, and deliver a pizza to House J. Assuming that you can cut across both lawns, how can you determine the parking location K that minimizes the distance $HK + KJ$? *(See Example 6.)*



28. **ATTENDING TO PRECISION** Use the numbers and symbols to create the glide reflection resulting in the image shown.



Translation: $(x, y) \rightarrow (\square, \square)$
Reflection: in $y = x$

1	2	3	
x	y	$+$	$-$

In Exercises 29–32, find point C on the x-axis so $AC + BC$ is a minimum.

29. $A(1, 4)$, $B(6, 1)$

30. $A(4, -5)$, $B(12, 3)$

31. $A(-8, 4)$, $B(-1, 3)$

32. $A(-1, 7)$, $B(5, -4)$

33. **MATHEMATICAL CONNECTIONS** The line $y = 3x + 2$ is reflected in the line $y = -1$. What is the equation of the image?

34. HOW DO YOU SEE IT? Use Figure A.

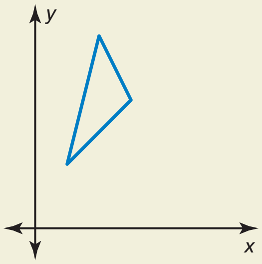


Figure A

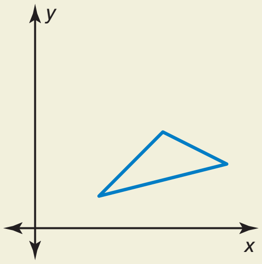


Figure 1

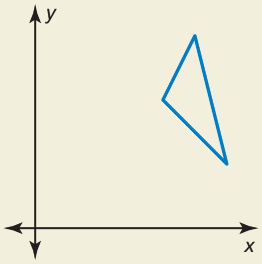


Figure 2

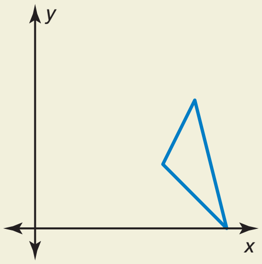


Figure 3

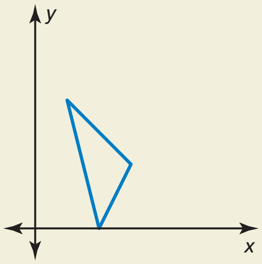


Figure 4

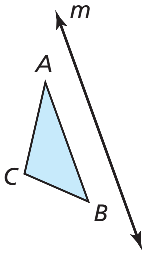
a. Which figure is a reflection of Figure A in the line $x = a$? Explain.

b. Which figure is a reflection of Figure A in the line $y = b$? Explain.

c. Which figure is a reflection of Figure A in the line $y = x$? Explain.

d. Is there a figure that represents a glide reflection? Explain your reasoning.

35. CONSTRUCTION Follow these steps to construct a reflection of $\triangle ABC$ in line m. Use a compass and straightedge.



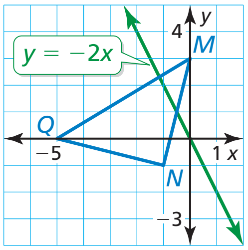
Step 1 Draw $\triangle ABC$ and line m.

Step 2 Use one compass setting to find two points that are equidistant from A on line m. Use the same compass setting to find a point on the other side of m that is the same distance from these two points. Label that point as A'.

Step 3 Repeat Step 2 to find points B' and C'. Draw $\triangle A'B'C'$.

36. USING TOOLS Use a reflective device to verify your construction in Exercise 35.

37. MATHEMATICAL CONNECTIONS Reflect $\triangle MNQ$ in the line $y = -2x$.



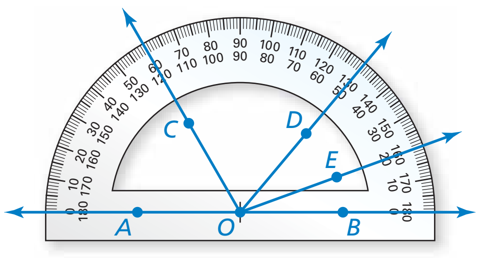
38. THOUGHT PROVOKING Is the composition of a translation and a reflection commutative? (In other words, do you obtain the same image regardless of the order in which you perform the transformations?) Justify your answer.

39. MATHEMATICAL CONNECTIONS Point $B'(1, 4)$ is the image of $B(3, 2)$ after a reflection in line c. Write an equation for line c.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use the diagram to find the angle measure. *(Section 1.5)*



40. $m\angle AOC$

41. $m\angle AOD$

42. $m\angle BOE$

43. $m\angle AOE$

44. $m\angle COD$

45. $m\angle EOD$

46. $m\angle COE$

47. $m\angle AOB$

48. $m\angle COB$

49. $m\angle BOD$

4.3 Rotations

Essential Question How can you rotate a figure in a coordinate plane?

EXPLORATION 1 Rotating a Triangle in a Coordinate Plane

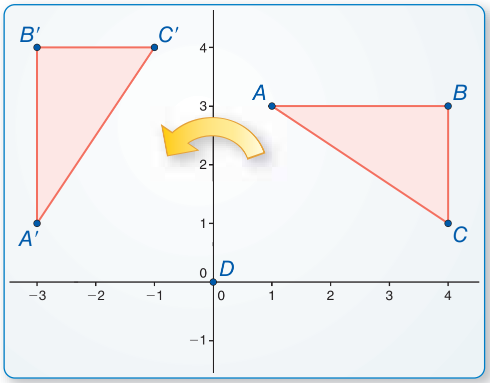
Work with a partner.

a. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

b. *Rotate* the triangle 90° counterclockwise about the origin to form $\triangle A'B'C'$.

c. What is the relationship between the coordinates of the vertices of $\triangle ABC$ and those of $\triangle A'B'C'$?

d. What do you observe about the side lengths and angle measures of the two triangles?



Sample

Points
$A(1, 3)$
$B(4, 3)$
$C(4, 1)$
$D(0, 0)$
Segments
$AB = 3$
$BC = 2$
$AC = 3.61$
Angles
$m\angle A = 33.69°$
$m\angle B = 90°$
$m\angle C = 56.31°$

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to use previously established results in constructing arguments.

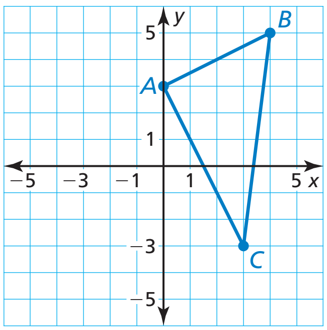
EXPLORATION 2 Rotating a Triangle in a Coordinate Plane

Work with a partner.

a. The point (x, y) is rotated 90° counterclockwise about the origin. Write a rule to determine the coordinates of the image of (x, y).

b. Use the rule you wrote in part (a) to rotate $\triangle ABC$ 90° counterclockwise about the origin. What are the coordinates of the vertices of the image, $\triangle A'B'C'$?

c. Draw $\triangle A'B'C'$. Are its side lengths the same as those of $\triangle ABC$? Justify your answer.



EXPLORATION 3 Rotating a Triangle in a Coordinate Plane

Work with a partner.

a. The point (x, y) is rotated 180° counterclockwise about the origin. Write a rule to determine the coordinates of the image of (x, y). Explain how you found the rule.

b. Use the rule you wrote in part (a) to rotate $\triangle ABC$ (from Exploration 2) 180° counterclockwise about the origin. What are the coordinates of the vertices of the image, $\triangle A'B'C'$?

Communicate Your Answer

4. How can you rotate a figure in a coordinate plane?

5. In Exploration 3, rotate $\triangle A'B'C'$ 180° counterclockwise about the origin. What are the coordinates of the vertices of the image, $\triangle A''B''C''$? How are these coordinates related to the coordinates of the vertices of the original triangle, $\triangle ABC$?

4.3 Lesson

Core Vocabulary

rotation, *p. 190*
center of rotation, *p. 190*
angle of rotation, *p. 190*
rotational symmetry, *p. 193*
center of symmetry, *p. 193*

What You Will Learn

- Perform rotations.
- Perform compositions with rotations.
- Identify rotational symmetry.

Performing Rotations

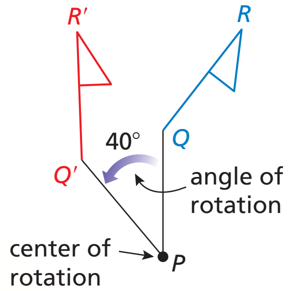
Core Concept

Rotations

A **rotation** is a transformation in which a figure is turned about a fixed point called the **center of rotation**. Rays drawn from the center of rotation to a point and its image form the **angle of rotation**.

A rotation about a point P through an angle of $x°$ maps every point Q in the plane to a point Q' so that one of the following properties is true.

- If Q is not the center of rotation P, then $QP = Q'P$ and $m\angle QPQ' = x°$, or
- If Q is the center of rotation P, then $Q = Q'$.



Direction of rotation

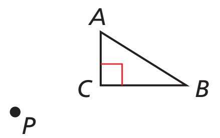
clockwise

counterclockwise

The figure above shows a 40° counterclockwise rotation. Rotations can be *clockwise* or *counterclockwise*. In this chapter, all rotations are counterclockwise unless otherwise noted.

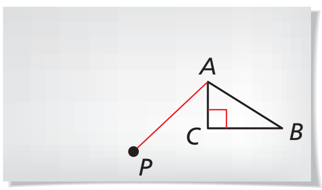
EXAMPLE 1 Drawing a Rotation

Draw a 120° rotation of $\triangle ABC$ about point P.

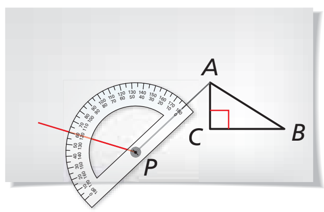


SOLUTION

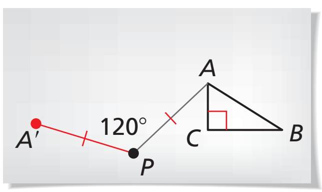
Step 1 Draw a segment from P to A.



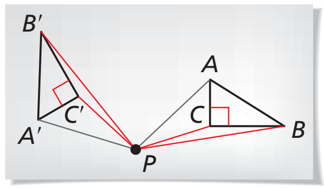
Step 2 Draw a ray to form a 120° angle with $\overline{PA}$.



Step 3 Draw A' so that $PA' = PA$.



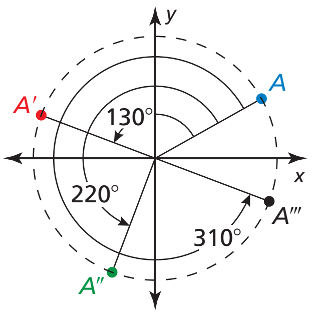
Step 4 Repeat Steps 1–3 for each vertex. Draw $\triangle A'B'C'$.



USING ROTATIONS

You can rotate a figure more than 360°. The effect, however, is the same as rotating the figure by the angle minus 360°.

You can rotate a figure more than 180°. The diagram shows rotations of point A 130°, 220°, and 310° about the origin. Notice that point A and its images all lie on the same circle. A rotation of 360° maps a figure onto itself.



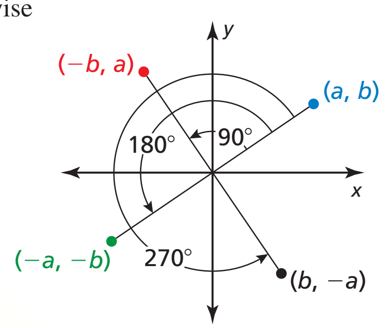
You can use coordinate rules to find the coordinates of a point after a rotation of 90°, 180°, or 270° about the origin.

Core Concept

Coordinate Rules for Rotations about the Origin

When a point (a, b) is rotated counterclockwise about the origin, the following are true.

- For a rotation of 90°, $(a, b) \rightarrow (-b, a)$.
- For a rotation of 180°, $(a, b) \rightarrow (-a, -b)$.
- For a rotation of 270°, $(a, b) \rightarrow (b, -a)$.



EXAMPLE 2 Rotating a Figure in the Coordinate Plane

Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$ and its image after a 270° rotation about the origin.

SOLUTION

Use the coordinate rule for a 270° rotation to find the coordinates of the vertices of the image. Then graph quadrilateral $RSTU$ and its image.

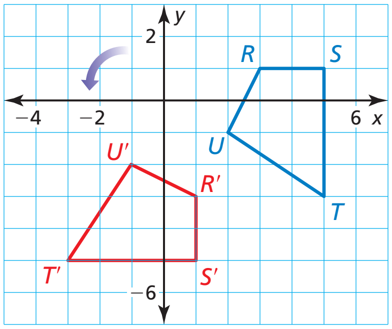
$$(a, b) \rightarrow (b, -a)$$

$$R(3, 1) \rightarrow R'(1, -3)$$

$$S(5, 1) \rightarrow S'(1, -5)$$

$$T(5, -3) \rightarrow T'(-3, -5)$$

$$U(2, -1) \rightarrow U'(-1, -2)$$



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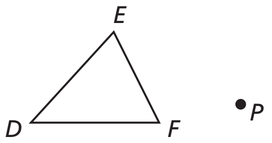
1. Trace $\triangle DEF$ and point P. Then draw a 50° rotation of $\triangle DEF$ about point P.



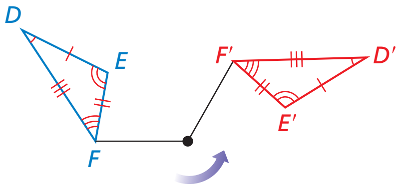
2. Graph $\triangle JKL$ with vertices $J(3, 0)$, $K(4, 3)$, and $L(6, 0)$ and its image after a 90° rotation about the origin.

Performing Compositions with Rotations

Postulate

Postulate 4.3 Rotation Postulate

A rotation is a rigid motion.



Because a rotation is a rigid motion, and a rigid motion preserves length and angle measure, the following statements are true for the rotation shown.

- $DE = D'E'$, $EF = E'F'$, $FD = F'D'$
- $m\angle D = m\angle D'$, $m\angle E = m\angle E'$, $m\angle F = m\angle F'$

Because a rotation is a rigid motion, the Composition Theorem (Theorem 4.1) guarantees that compositions of rotations and other rigid motions, such as translations and reflections, are rigid motions.

EXAMPLE 3 Performing a Composition

Graph $\overline{RS}$ with endpoints $R(1, -3)$ and $S(2, -6)$ and its image after the composition.

Reflection: in the y-axis

Rotation: 90° about the origin

COMMON ERROR

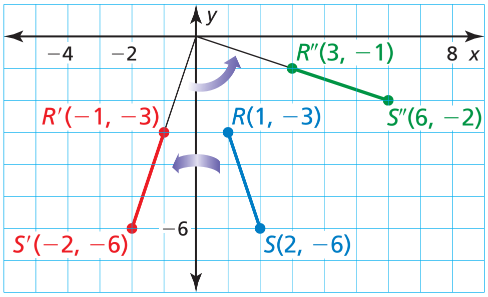
Unless you are told otherwise, perform the transformations in the order given.

SOLUTION

Step 1 Graph $\overline{RS}$.

Step 2 Reflect $\overline{RS}$ in the y-axis. $\overline{R'S'}$ has endpoints $R'(-1, -3)$ and $S'(-2, -6)$.

Step 3 Rotate $\overline{R'S'}$ 90° about the origin. $\overline{R''S''}$ has endpoints $R''(3, -1)$ and $S''(6, -2)$.



Monitoring Progress

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3. Graph $\overline{RS}$ from Example 3. Perform the rotation first, followed by the reflection. Does the order of the transformations matter? Explain.

4. **WHAT IF?** In Example 3, $\overline{RS}$ is reflected in the x-axis and rotated 180° about the origin. Graph $\overline{RS}$ and its image after the composition.

5. Graph $\overline{AB}$ with endpoints $A(-4, 4)$ and $B(-1, 7)$ and its image after the composition.

Translation: $(x, y) \rightarrow (x - 2, y - 1)$

Rotation: 90° about the origin

6. Graph $\triangle TUV$ with vertices $T(1, 2)$, $U(3, 5)$, and $V(6, 3)$ and its image after the composition.

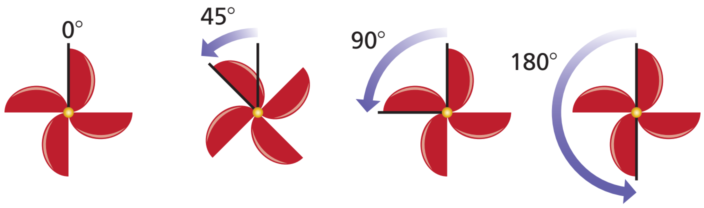
Rotation: 180° about the origin

Reflection: in the x-axis

Identifying Rotational Symmetry

A figure in the plane has **rotational symmetry** when the figure can be mapped onto itself by a rotation of 180° or less about the center of the figure. This point is the **center of symmetry**. Note that the rotation can be either clockwise or counterclockwise.

For example, the figure below has rotational symmetry, because a rotation of either 90° or 180° maps the figure onto itself (although a rotation of 45° does not).

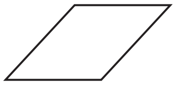


The figure above also has *point symmetry*, which is 180° rotational symmetry.

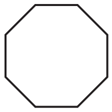
EXAMPLE 4 Identifying Rotational Symmetry

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

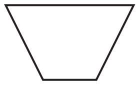
a. parallelogram



b. regular octagon

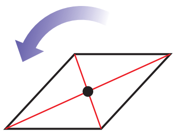


c. trapezoid

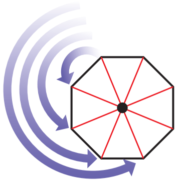


SOLUTION

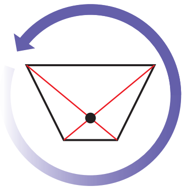
a. The parallelogram has rotational symmetry. The center is the intersection of the diagonals. A 180° rotation about the center maps the parallelogram onto itself.



b. The regular octagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of 45°, 90°, 135°, or 180° about the center all map the octagon onto itself.



c. The trapezoid does not have rotational symmetry because no rotation of 180° or less maps the trapezoid onto itself.



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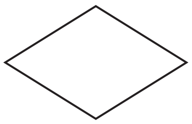
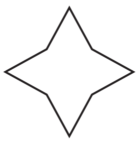
Determine whether the figure has rotational symmetry. If so, describe any rotations that map the figure onto itself.

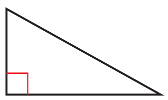
7. rhombus



8. octagon



9. right triangle



4.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** When a point (a, b) is rotated counterclockwise about the origin, $(a, b) \rightarrow (b, -a)$ is the result of a rotation of ______.

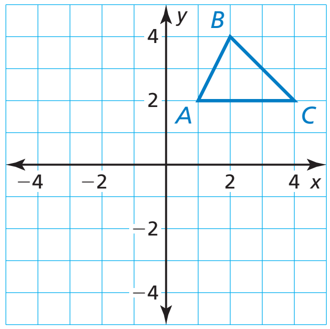
2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

What are the coordinates of the vertices of the image after a 90° counterclockwise rotation about the origin?

What are the coordinates of the vertices of the image after a 270° clockwise rotation about the origin?

What are the coordinates of the vertices of the image after turning the figure 90° to the left about the origin?

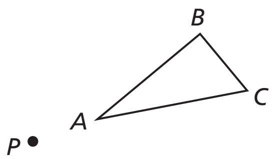
What are the coordinates of the vertices of the image after a 270° counterclockwise rotation about the origin?



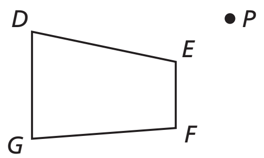
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, trace the polygon and point *P*. Then draw a rotation of the polygon about point *P* using the given number of degrees. *(See Example 1.)*

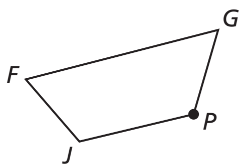
3. 30°



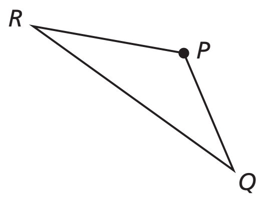
4. 80°



5. 150°

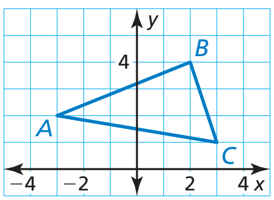


6. 130°

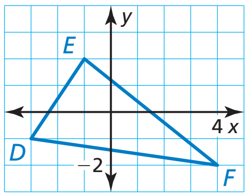


In Exercises 7–10, graph the polygon and its image after a rotation of the given number of degrees about the origin. *(See Example 2.)*

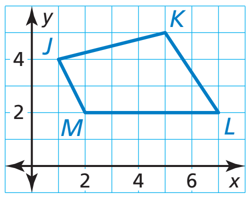
7. 90°



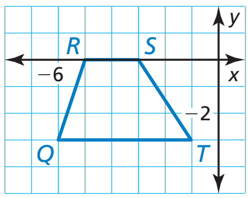
8. 180°



9. 180°



10. 270°



In Exercises 11–14, graph $\overline{XY}$ with endpoints $X(-3, 1)$ and $Y(4, -5)$ and its image after the composition. *(See Example 3.)*

11. **Translation:** $(x, y) \rightarrow (x, y + 2)$
Rotation: 90° about the origin

12. **Rotation:** 180° about the origin
Translation: $(x, y) \rightarrow (x - 1, y + 1)$

13. **Rotation:** 270° about the origin
Reflection: in the y-axis

14. **Reflection:** in the line $y = x$
Rotation: 180° about the origin

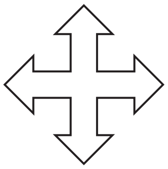
In Exercises 15 and 16, graph $\triangle LMN$ with vertices $L(1, 6)$, $M(-2, 4)$, and $N(3, 2)$ and its image after the composition. *(See Example 3.)*

15. **Rotation:** 90° about the origin
Translation: $(x, y) \rightarrow (x - 3, y + 2)$

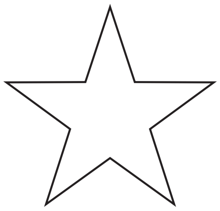
16. **Reflection:** in the x-axis
Rotation: 270° about the origin

In Exercises 17–20, determine whether the figure has rotational symmetry. If so, describe any rotations that map the figure onto itself. *(See Example 4.)*

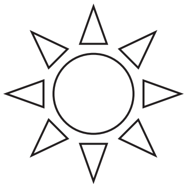
17.



18.



19.

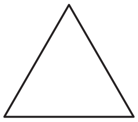


20.

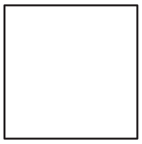
REPEATED REASONING **In Exercises 21–24, select the angles of rotational symmetry for the regular polygon. Select all that apply.**

Ⓐ 30° Ⓑ 45° Ⓒ 60° Ⓓ 72°
Ⓔ 90° Ⓕ 120° Ⓖ 144° Ⓗ 180°

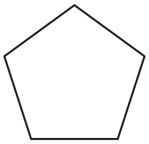
21.



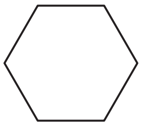
22.



23.



24.



ERROR ANALYSIS **In Exercises 25 and 26, the endpoints of $\overline{CD}$ are $C(-1, 1)$ and $D(2, 3)$. Describe and correct the error in finding the coordinates of the vertices of the image after a rotation of 270° about the origin.**

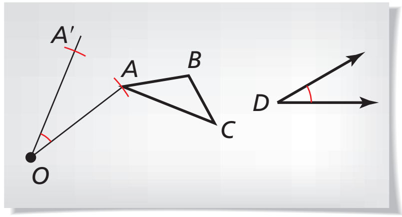
25.

✗ $C(-1, 1) \rightarrow C'(-1, -1)$
$D(2, 3) \rightarrow D'(2, -3)$

26.

✗ $C(-1, 1) \rightarrow C'(1, -1)$
$D(2, 3) \rightarrow D'(3, 2)$

27. **CONSTRUCTION** Follow these steps to construct a rotation of $\triangle ABC$ by angle D around a point O. Use a compass and straightedge.



Step 1 Draw $\triangle ABC$, $\angle D$, and O, the center of rotation.

Step 2 Draw $\overline{OA}$. Use the construction for copying an angle to copy $\angle D$ at O, as shown. Then use distance OA and center O to find A'.

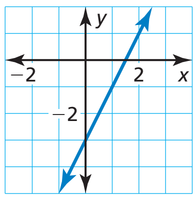
Step 3 Repeat Step 2 to find points B' and C'. Draw $\triangle A'B'C'$.

28. **REASONING** You enter the revolving door at a hotel.

a. You rotate the door 180°. What does this mean in the context of the situation? Explain.

b. You rotate the door 360°. What does this mean in the context of the situation? Explain.

29. **MATHEMATICAL CONNECTIONS** Use the graph of $y = 2x - 3$.



a. Rotate the line 90°, 180°, 270°, and 360° about the origin. Write the equation of the line for each image. Describe the relationship between the equation of the preimage and the equation of each image.

b. Do you think that the relationships you described in part (a) are true for any line that is not vertical or horizontal? Explain your reasoning.

30. **MAKING AN ARGUMENT** Your friend claims that rotating a figure by 180° is the same as reflecting a figure in the y-axis and then reflecting it in the x-axis. Is your friend correct? Explain your reasoning.

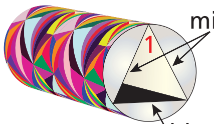
31. DRAWING CONCLUSIONS A figure only has point symmetry. How many rotations that map the figure onto itself can be performed before it is back where it started?

32. ANALYZING RELATIONSHIPS Is it possible for a figure to have 90° rotational symmetry but not 180° rotational symmetry? Explain your reasoning.

33. ANALYZING RELATIONSHIPS Is it possible for a figure to have 180° rotational symmetry but not 90° rotational symmetry? Explain your reasoning.

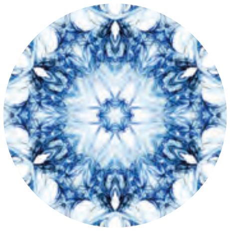
34. THOUGHT PROVOKING Can rotations of 90°, 180°, 270°, and 360° be written as the composition of two reflections? Justify your answer.

35. USING AN EQUATION Inside a kaleidoscope, two mirrors are placed next to each other to form a V. The angle between the mirrors determines the number of lines of symmetry in the image. Use the formula $n(m\angle 1) = 180°$ to find the measure of $\angle 1$, the angle between the mirrors, for the number n of lines of symmetry.



a.

b.



36. REASONING Use the coordinate rules for counterclockwise rotations about the origin to write coordinate rules for clockwise rotations of 90°, 180°, or 270° about the origin.

37. USING STRUCTURE $\triangle XYZ$ has vertices $X(2, 5)$, $Y(3, 1)$, and $Z(0, 2)$. Rotate $\triangle XYZ$ 90° about the point $P(-2, -1)$.

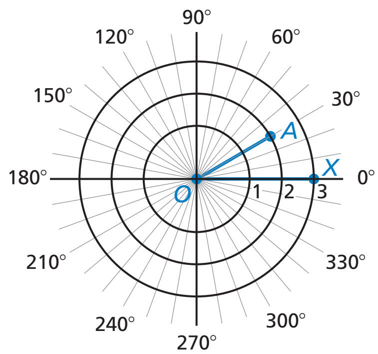
38. HOW DO YOU SEE IT? You are finishing the puzzle. The remaining two pieces both have rotational symmetry.

a. Describe the rotational symmetry of Piece 1 and of Piece 2.

b. You pick up Piece 1. How many different ways can it fit in the puzzle?

c. Before putting Piece 1 into the puzzle, you connect it to Piece 2. Now how many ways can it fit in the puzzle? Explain.

39. USING STRUCTURE A polar coordinate system locates a point in a plane by its distance from the origin O and by the measure of an angle with its vertex at the origin. For example, the point $A(2, 30°)$ is 2 units from the origin and $m\angle XOA = 30°$. What are the polar coordinates of the image of point A after a 90° rotation? a 180° rotation? a 270° rotation? Explain.

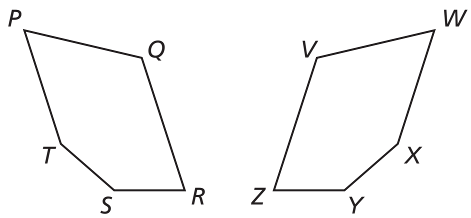


Maintaining Mathematical Proficiency

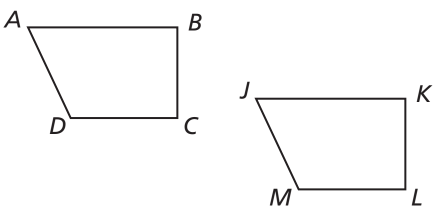
Reviewing what you learned in previous grades and lessons

The figures are congruent. Name the corresponding angles and the corresponding sides. *(Skills Review Handbook)*

40.



41.



4.1–4.3 What Did You Learn?

Core Vocabulary

vector, *p. 174*
initial point, *p. 174*
terminal point, *p. 174*
horizontal component, *p. 174*
vertical component, *p. 174*
component form, *p. 174*
transformation, *p. 174*
image, *p. 174*
preimage, *p. 174*
translation, *p. 174*
rigid motion, *p. 176*
composition of transformations, *p. 176*
reflection, *p. 182*
line of reflection, *p. 182*
glide reflection, *p. 184*
line symmetry, *p. 185*
line of symmetry, *p. 185*
rotation, *p. 190*
center of rotation, *p. 190*
angle of rotation, *p. 190*
rotational symmetry, *p. 193*
center of symmetry, *p. 193*

Core Concepts

Section 4.1

Vectors, *p. 174*
Translations, *p. 174*
Postulate 4.1 Translation Postulate, *p. 176*
Theorem 4.1 Composition Theorem, *p. 176*

Section 4.2

Reflections, *p. 182*
Coordinate Rules for Reflections, *p. 183*
Postulate 4.2 Reflection Postulate, *p. 184*
Line Symmetry, *p. 185*

Section 4.3

Rotations, *p. 190*
Coordinate Rules for Rotations about the Origin, *p. 191*
Postulate 4.3 Rotation Postulate, *p. 192*
Rotational Symmetry, *p. 193*

Mathematical Practices

1. How could you determine whether your results make sense in Exercise 26 on page 179?
2. State the meaning of the numbers and symbols you chose in Exercise 28 on page 187.
3. Describe the steps you would take to arrive at the answer to Exercise 29 part (a) on page 195.

Study Skills

Keeping a Positive Attitude

Ever feel frustrated or overwhelmed by math? You're not alone. Just take a deep breath and assess the situation. Try to find a productive study environment, review your notes and examples in the textbook, and ask your teacher or peers for help.

4.1–4.3 Quiz

Graph quadrilateral *ABCD* with vertices $A(-4, 1)$, $B(-3, 3)$, $C(0, 1)$, and $D(-2, 0)$ and its image after the translation. *(Section 4.1)*

1. $(x, y) \rightarrow (x + 4, y - 2)$

2. $(x, y) \rightarrow (x - 1, y - 5)$

3. $(x, y) \rightarrow (x + 3, y + 6)$

Graph the polygon with the given vertices and its image after a reflection in the given line. *(Section 4.2)*

4. $A(-5, 6)$, $B(-7, 8)$, $C(-3, 11)$; x-axis

5. $D(-5, -1)$, $E(-2, 1)$, $F(-1, -3)$; $y = x$

6. $J(-1, 4)$, $K(2, 5)$, $L(5, 2)$, $M(4, -1)$; $x = 3$

7. $P(2, -4)$, $Q(6, -1)$, $R(9, -4)$, $S(6, -6)$; $y = -2$

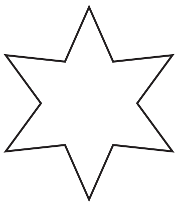
Graph $\triangle ABC$ with vertices $A(2, -1)$, $B(5, 2)$, and $C(8, -2)$ and its image after the glide reflection. *(Section 4.2)*

8. **Translation:** $(x, y) \rightarrow (x, y + 6)$
Reflection: in the y-axis

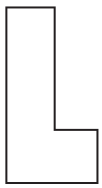
9. **Translation:** $(x, y) \rightarrow (x - 9, y)$
Reflection: in the line $y = 1$

Determine the number of lines of symmetry for the figure. *(Section 4.2)*

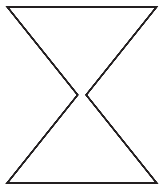
10.



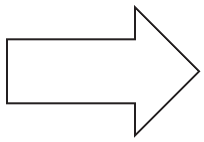
11.



12.

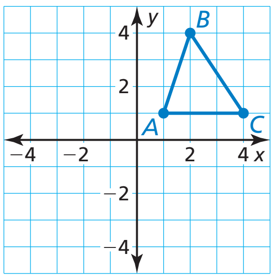


13.

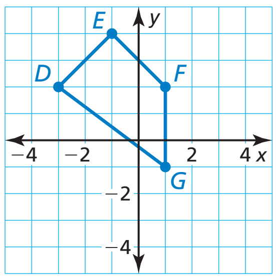


Graph the polygon and its image after a rotation of the given number of degrees about the origin. *(Section 4.3)*

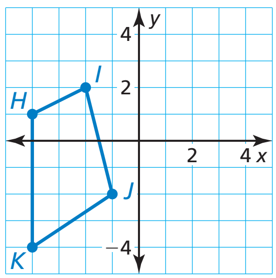
14. 90°



15. 270°



16. 180°

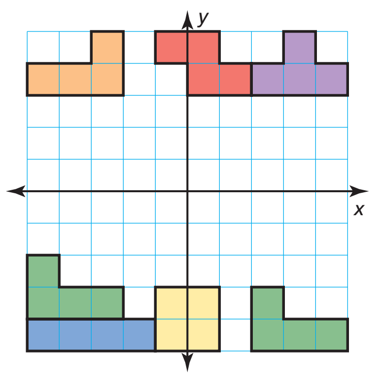


Graph $\triangle LMN$ with vertices $L(-3, -2)$, $M(-1, 1)$, and $N(2, -3)$ and its image after the composition. *(Sections 4.1–4.3)*

17. **Translation:** $(x, y) \rightarrow (x - 4, y + 3)$
Rotation: 180° about the origin

18. **Rotation:** 90° about the origin
Reflection: in the y-axis

19. The figure shows a game in which the object is to create solid rows using the pieces given. Using only translations and rotations, describe the transformations for each piece at the top that will form two solid rows at the bottom. *(Section 4.1 and Section 4.3)*



4.4 Congruence and Transformations

Essential Question What conjectures can you make about a figure reflected in two lines?

EXPLORATION 1 Reflections in Parallel Lines

Work with a partner. Use dynamic geometry software to draw any scalene triangle and label it $\triangle ABC$.

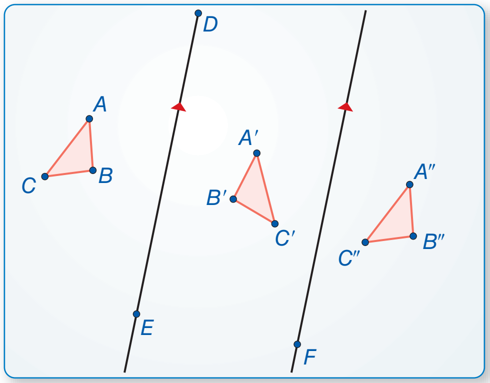
a. Draw any line $\overleftrightarrow{DE}$. Reflect $\triangle ABC$ in $\overleftrightarrow{DE}$ to form $\triangle A'B'C'$.

b. Draw a line parallel to $\overleftrightarrow{DE}$. Reflect $\triangle A'B'C'$ in the new line to form $\triangle A''B''C''$.

c. Draw the line through point A that is perpendicular to $\overleftrightarrow{DE}$. What do you notice?

d. Find the distance between points A and A''. Find the distance between the two parallel lines. What do you notice?

Sample



e. Hide $\triangle A'B'C'$. Is there a single transformation that maps $\triangle ABC$ to $\triangle A''B''C''$? Explain.

f. Make conjectures based on your answers in parts (c)–(e). Test your conjectures by changing $\triangle ABC$ and the parallel lines.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and justify your conclusions.

EXPLORATION 2 Reflections in Intersecting Lines

Work with a partner. Use dynamic geometry software to draw any scalene triangle and label it $\triangle ABC$.

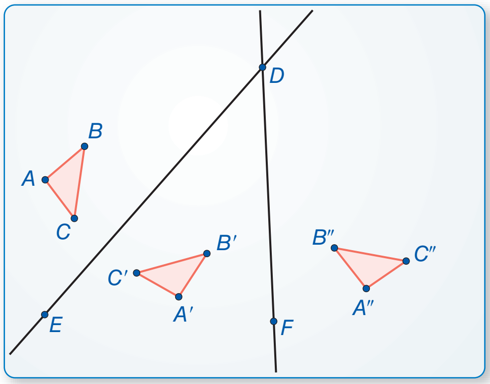
a. Draw any line $\overleftrightarrow{DE}$. Reflect $\triangle ABC$ in $\overleftrightarrow{DE}$ to form $\triangle A'B'C'$.

b. Draw any line $\overleftrightarrow{DF}$ so that angle EDF is less than or equal to 90°. Reflect $\triangle A'B'C'$ in $\overleftrightarrow{DF}$ to form $\triangle A''B''C''$.

c. Find the measure of $\angle EDF$. Rotate $\triangle ABC$ counterclockwise about point D using an angle twice the measure of $\angle EDF$.

d. Make a conjecture about a figure reflected in two intersecting lines. Test your conjecture by changing $\triangle ABC$ and the lines.

Sample



Communicate Your Answer

3. What conjectures can you make about a figure reflected in two lines?

4. Point Q is reflected in two parallel lines, $\overleftrightarrow{GH}$ and $\overleftrightarrow{JK}$, to form Q' and Q''. The distance from $\overleftrightarrow{GH}$ to $\overleftrightarrow{JK}$ is 3.2 inches. What is the distance QQ''?

4.4 Lesson

Core Vocabulary

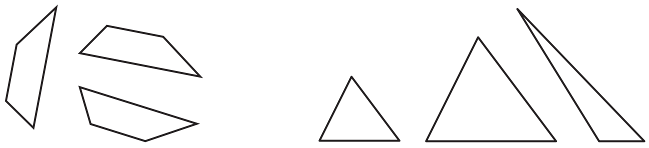
congruent figures, *p. 200*
congruence transformation, *p. 201*

What You Will Learn

- Identify congruent figures.
- Describe congruence transformations.
- Use theorems about congruence transformations.

Identifying Congruent Figures

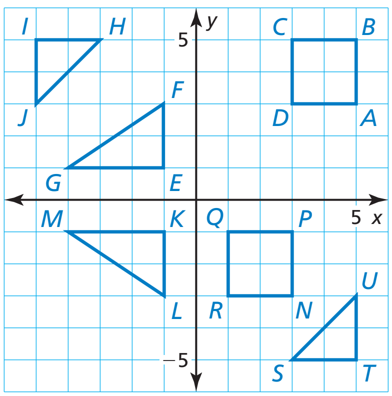
Two geometric figures are **congruent figures** if and only if there is a rigid motion or a composition of rigid motions that maps one of the figures onto the other. Congruent figures have the same size and shape.



You can identify congruent figures in the coordinate plane by identifying the rigid motion or composition of rigid motions that maps one of the figures onto the other. Recall from Postulates 4.1–4.3 and Theorem 4.1 that translations, reflections, rotations, and compositions of these transformations are rigid motions.

EXAMPLE 1 Identifying Congruent Figures

Identify any congruent figures in the coordinate plane. Explain.



SOLUTION

Square *NPQR* is a translation of square *ABCD* 2 units left and 6 units down. So, square *ABCD* and square *NPQR* are congruent.

$\triangle KLM$ is a reflection of $\triangle EFG$ in the x-axis. So, $\triangle EFG$ and $\triangle KLM$ are congruent.

$\triangle STU$ is a 180° rotation of $\triangle HIJ$. So, $\triangle HIJ$ and $\triangle STU$ are congruent.

Monitoring Progress

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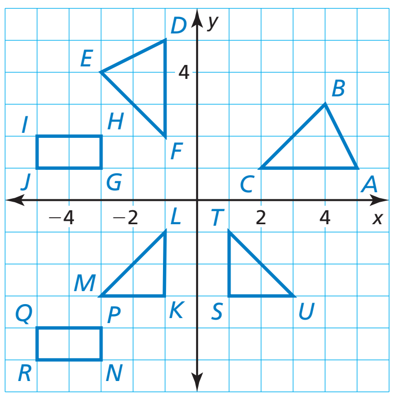
1. Identify any congruent figures in the coordinate plane. Explain.



Congruence Transformations

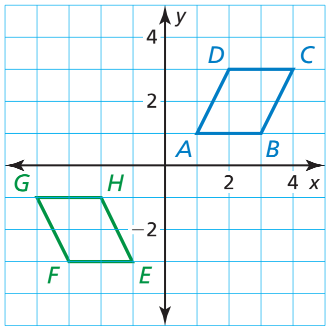
Another name for a rigid motion or a combination of rigid motions is a **congruence transformation** because the preimage and image are congruent. The terms "rigid motion" and "congruence transformation" are interchangeable.

READING

You can read the notation ▱*ABCD* as "parallelogram *A, B, C, D*."

EXAMPLE 2 **Describing a Congruence Transformation**

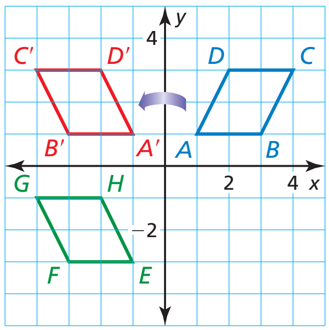
Describe a congruence transformation that maps ▱$ABCD$ to ▱$EFGH$.



SOLUTION

The two vertical sides of ▱$ABCD$ rise from left to right, and the two vertical sides of ▱$EFGH$ fall from left to right. If you reflect ▱$ABCD$ in the y-axis, as shown, then the image, ▱$A'B'C'D'$, will have the same orientation as ▱$EFGH$.

Then you can map ▱$A'B'C'D'$ to ▱$EFGH$ using a translation of 4 units down.



▶ So, a congruence transformation that maps ▱$ABCD$ to ▱$EFGH$ is a reflection in the y-axis followed by a translation of 4 units down.

Monitoring Progress

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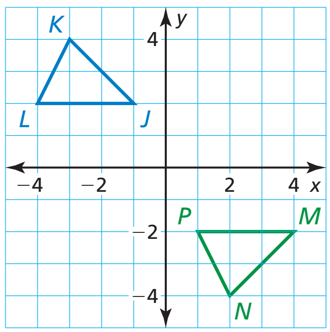
2. In Example 2, describe another congruence transformation that maps ▱$ABCD$ to ▱$EFGH$.

3. Describe a congruence transformation that maps $\triangle JKL$ to $\triangle MNP$.



Using Theorems about Congruence Transformations

Compositions of two reflections result in either a translation or a rotation. A composition of two reflections in parallel lines results in a translation, as described in the following theorem.

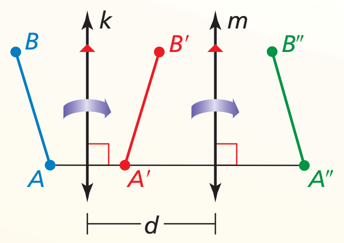
Theorem

Theorem 4.2 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If A'' is the image of A, then

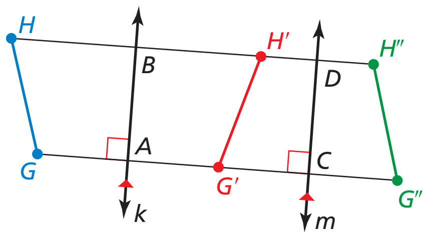
1. $\overline{AA''}$ is perpendicular to k and m, and
2. $AA'' = 2d$, where d is the distance between k and m.



Proof Ex. 31, p. 206

EXAMPLE 3 Using the Reflections in Parallel Lines Theorem

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $DH'' = 4$.



a. Name any segments congruent to each segment: $\overline{GH}$, $\overline{HB}$, and $\overline{GA}$.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?

SOLUTION

a. $\overline{GH} \cong \overline{G'H'}$, and $\overline{GH} \cong \overline{G''H''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.

b. Yes, $AC = BD$ because $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m. So, $\overline{BD}$ and $\overline{AC}$ are opposite sides of a rectangle.

c. By the properties of reflections, $H'B = 9$ and $H'D = 4$. The Reflections in Parallel Lines Theorem implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4) = 26$ units.

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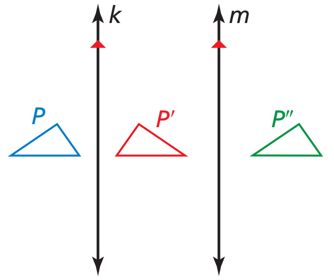
Use the figure. The distance between line k and line m is 1.6 centimeters.



4. The preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure to the green figure.

5. What is the relationship between $\overline{PP'}$ and line k? Explain.

6. What is the distance between P and P''?

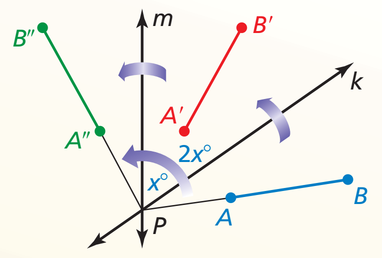
A composition of two reflections in intersecting lines results in a rotation, as described in the following theorem.

Theorem

Theorem 4.3 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P, then a reflection in line k followed by a reflection in line m is the same as a rotation about point P.

The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by lines k and m.

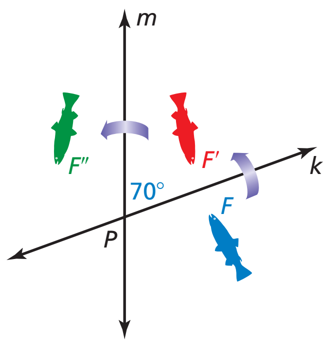


$m\angle BPB'' = 2x°$

Proof Ex. 31, p. 250

EXAMPLE 4 Using the Reflections in Intersecting Lines Theorem

In the diagram, the figure is reflected in line k. The image is then reflected in line m. Describe a single transformation that maps F to F''.



SOLUTION

By the Reflections in Intersecting Lines Theorem, a reflection in line k followed by a reflection in line m is the same as a rotation about point P. The measure of the acute angle formed between lines k and m is 70°. So, by the Reflections in Intersecting Lines Theorem, the angle of rotation is $2(70°) = 140°$. A single transformation that maps F to F'' is a 140° rotation about point P.

▶ You can check that this is correct by tracing lines k and m and point F, then rotating the point 140°.

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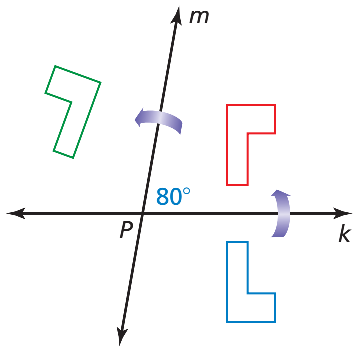
7. In the diagram, the preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure onto the green figure.

8. A rotation of 76° maps C to C'. To map C to C' using two reflections, what is the measure of the angle formed by the intersecting lines of reflection?



4.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

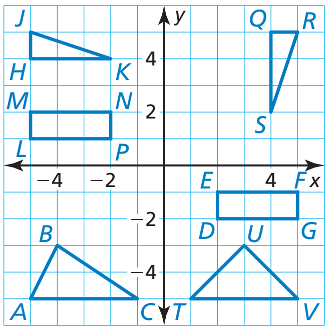
1. **COMPLETE THE SENTENCE** Two geometric figures are __________ if and only if there is a rigid motion or a composition of rigid motions that moves one of the figures onto the other.

2. **VOCABULARY** Why is the term *congruence transformation* used to refer to a rigid motion?

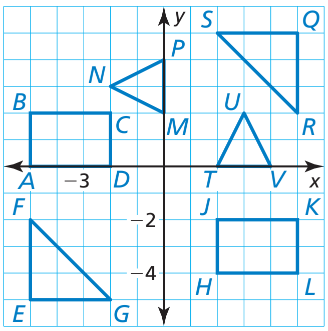
Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, identify any congruent figures in the coordinate plane. Explain. *(See Example 1.)*

3.

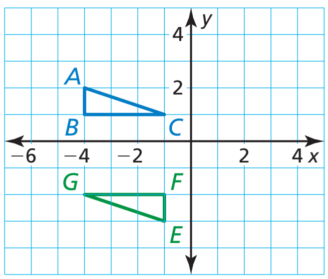


4.

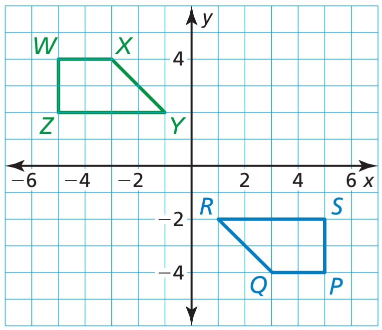


In Exercises 5 and 6, describe a congruence transformation that maps the blue preimage to the green image. *(See Example 2.)*

5.



6.



In Exercises 7–10, determine whether the polygons with the given vertices are congruent. Use transformations to explain your reasoning.

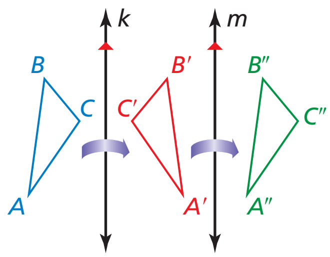
7. $Q(2, 4)$, $R(5, 4)$, $S(4, 1)$ and $T(6, 4)$, $U(9, 4)$, $V(8, 1)$

8. $W(-3, 1)$, $X(2, 1)$, $Y(4, -4)$, $Z(-5, -4)$ and $C(-1, -3)$, $D(-1, 2)$, $E(4, 4)$, $F(4, -5)$

9. $J(1, 1)$, $K(3, 2)$, $L(4, 1)$ and $M(6, 1)$, $N(5, 2)$, $P(2, 1)$

10. $A(0, 0)$, $B(1, 2)$, $C(4, 2)$, $D(3, 0)$ and $E(0, -5)$, $F(-1, -3)$, $G(-4, -3)$, $H(-3, -5)$

In Exercises 11–14, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m. *(See Example 3.)*



11. A translation maps $\triangle ABC$ onto which triangle?

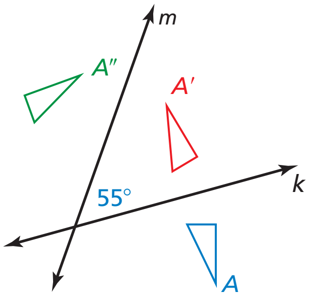
12. Which lines are perpendicular to $\overline{AA''}$?

13. If the distance between k and m is 2.6 inches, what is the length of $\overline{CC''}$?

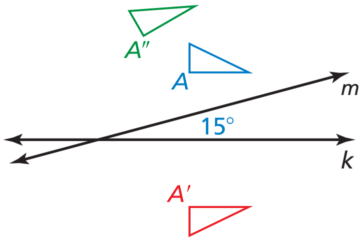
14. Is the distance from B' to m the same as the distance from B'' to m? Explain.

In Exercises 15 and 16, find the angle of rotation that maps A onto A''. *(See Example 4.)*

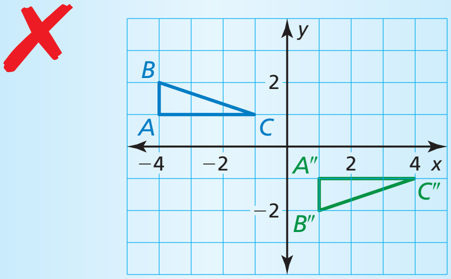
15.



16.

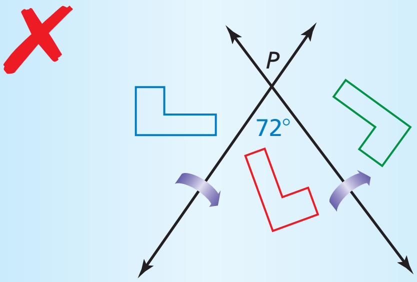


17. ERROR ANALYSIS Describe and correct the error in describing the congruence transformation.



$\triangle ABC$ is mapped to $\triangle A''B''C''$ by a translation 3 units down and a reflection in the y-axis.

18. ERROR ANALYSIS Describe and correct the error in using the Reflections in Intersecting Lines Theorem (Theorem 4.3).



A 72° rotation about point P maps the blue image to the green image.

In Exercises 19–22, find the measure of the acute or right angle formed by intersecting lines so that C can be mapped to C' using two reflections.

19. A rotation of 84° maps C to C'.

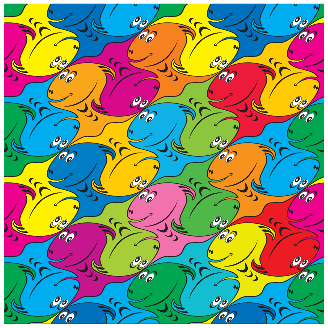
20. A rotation of 24° maps C to C'.

21. The rotation $(x, y) \rightarrow (-x, -y)$ maps C to C'.

22. The rotation $(x, y) \rightarrow (y, -x)$ maps C to C'.

23. REASONING Use the Reflections in Parallel Lines Theorem (Theorem 4.2) to explain how you can make a glide reflection using three reflections. How are the lines of reflection related?

24. DRAWING CONCLUSIONS The pattern shown is called a *tessellation*.



a. What transformations did the artist use when creating this tessellation?

b. Are the individual figures in the tessellation congruent? Explain your reasoning.

CRITICAL THINKING In Exercises 25–28, tell whether the statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

25. A congruence transformation changes the size of a figure.

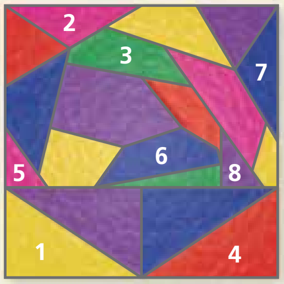
26. If two figures are congruent, then there is a rigid motion or a composition of rigid motions that maps one figure onto the other.

27. The composition of two reflections results in the same image as a rotation.

28. A translation results in the same image as the composition of two reflections.

29. REASONING During a presentation, a marketing representative uses a projector so everyone in the auditorium can view the advertisement. Is this projection a congruence transformation? Explain your reasoning.

30. **HOW DO YOU SEE IT?** What type of congruence transformation can be used to verify each statement about the stained glass window?



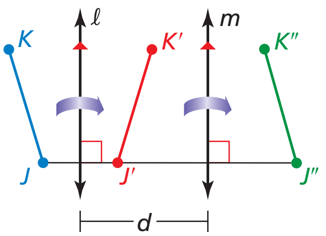
a. Triangle 5 is congruent to Triangle 8.

b. Triangle 1 is congruent to Triangle 4.

c. Triangle 2 is congruent to Triangle 7.

d. Pentagon 3 is congruent to Pentagon 6.

31. **PROVING A THEOREM** Prove the Reflections in Parallel Lines Theorem (Theorem 4.2).



Given A reflection in line ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in line m maps $\overline{J'K'}$ to $\overline{J''K''}$, and $\ell \parallel m$.

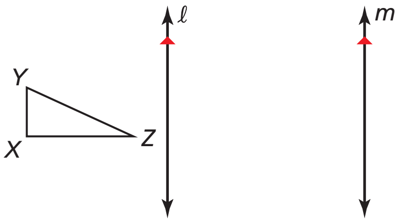
Prove **a.** $\overline{KK''}$ is perpendicular to ℓ and m.

b. $KK'' = 2d$, where d is the distance between ℓ and m.

32. **THOUGHT PROVOKING** A *tessellation* is the covering of a plane with congruent figures so that there are no gaps or overlaps (see Exercise 24). Draw a tessellation that involves two or more types of transformations. Describe the transformations that are used to create the tessellation.

33. **MAKING AN ARGUMENT** $\overline{PQ}$, with endpoints $P(1, 3)$ and $Q(3, 2)$, is reflected in the y-axis. The image $\overline{P'Q'}$ is then reflected in the x-axis to produce the image $\overline{P''Q''}$. One classmate says that $\overline{PQ}$ is mapped to $\overline{P''Q''}$ by the translation $(x, y) \rightarrow (x - 4, y - 5)$. Another classmate says that $\overline{PQ}$ is mapped to $\overline{P''Q''}$ by a $(2 \cdot 90)°$, or $180°$, rotation about the origin. Which classmate is correct? Explain your reasoning.

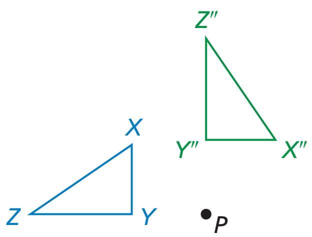
34. **CRITICAL THINKING** Does the order of reflections for a composition of two reflections in parallel lines matter? For example, is reflecting $\triangle XYZ$ in line ℓ and then its image in line m the same as reflecting $\triangle XYZ$ in line m and then its image in line ℓ?



CONSTRUCTION In Exercises 35 and 36, copy the figure. Then use a compass and straightedge to construct two lines of reflection that produce a composition of reflections resulting in the same image as the given transformation.

35. **Translation: $\triangle ABC \rightarrow \triangle A''B''C''$**

36. **Rotation about P: $\triangle XYZ \rightarrow \triangle X''Y''Z''$**



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Skills Review Handbook)*

37. $5x + 16 = -3x$

38. $12 + 6m = 2m$

39. $4b + 8 = 6b - 4$

40. $7w - 9 = 13 - 4w$

41. $7(2n + 11) = 4n$

42. $-2(8 - y) = -6y$

43. Last year, the track team's yard sale earned \$500. This year, the yard sale earned \$625. What is the percent of increase? *(Skills Review Handbook)*

4.5 Dilations

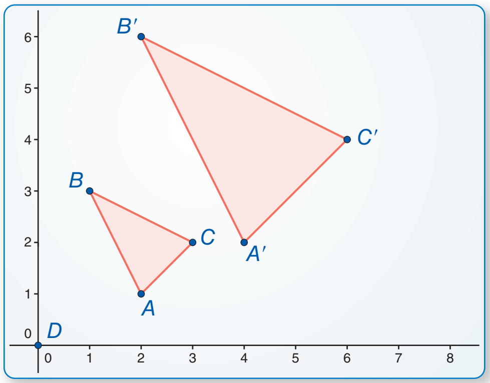
Essential Question

What does it mean to dilate a figure?

EXPLORATION 1 Dilating a Triangle in a Coordinate Plane

Work with a partner. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

a. *Dilate* $\triangle ABC$ using a *scale factor* of 2 and a *center of dilation* at the origin to form $\triangle A'B'C'$. Compare the coordinates, side lengths, and angle measures of $\triangle ABC$ and $\triangle A'B'C'$.



Sample

Points
$A(2, 1)$
$B(1, 3)$
$C(3, 2)$
Segments
$AB = 2.24$
$BC = 2.24$
$AC = 1.41$
Angles
$m\angle A = 71.57°$
$m\angle B = 36.87°$
$m\angle C = 71.57°$

b. Repeat part (a) using a *scale factor* of $\frac{1}{2}$.

c. What do the results of parts (a) and (b) suggest about the coordinates, side lengths, and angle measures of the image of $\triangle ABC$ after a dilation with a scale factor of k?

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Dilating Lines in a Coordinate Plane

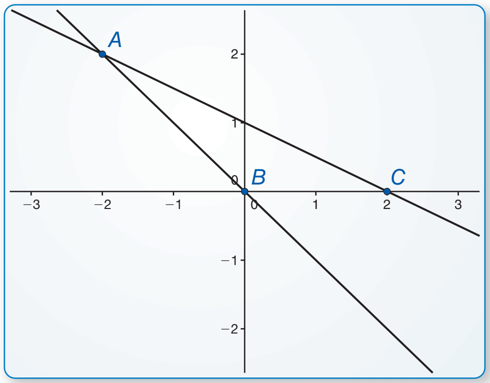
Work with a partner. Use dynamic geometry software to draw $\overleftrightarrow{AB}$ that passes through the origin and $\overleftrightarrow{AC}$ that does not pass through the origin.

a. *Dilate* $\overleftrightarrow{AB}$ using a *scale factor* of 3 and a *center of dilation* at the origin. Describe the image.

b. *Dilate* $\overleftrightarrow{AC}$ using a *scale factor* of 3 and a *center of dilation* at the origin. Describe the image.

c. Repeat parts (a) and (b) using a scale factor of $\frac{1}{4}$.

d. What do you notice about dilations of lines passing through the center of dilation and dilations of lines not passing through the center of dilation?



Sample

Points
$A(-2, 2)$
$B(0, 0)$
$C(2, 0)$

Lines
$x + y = 0$
$x + 2y = 2$

Communicate Your Answer

3. What does it mean to dilate a figure?

4. Repeat Exploration 1 using a center of dilation at a point other than the origin.

4.5 Lesson

Core Vocabulary

dilation, *p. 208*
center of dilation, *p. 208*
scale factor, *p. 208*
enlargement, *p. 208*
reduction, *p. 208*

What You Will Learn

- Identify and perform dilations.
- Solve real-life problems involving scale factors and dilations.

Identifying and Performing Dilations

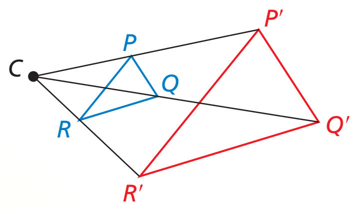
Core Concept

Dilations

A **dilation** is a transformation in which a figure is enlarged or reduced with respect to a fixed point C called the **center of dilation** and a **scale factor** k, which is the ratio of the lengths of the corresponding sides of the image and the preimage.

A dilation with center of dilation C and scale factor k maps every point P in a figure to a point P' so that the following are true.

- If P is the center point C, then $P = P'$.
- If P is not the center point C, then the image point P' lies on $\overrightarrow{CP}$. The scale factor k is a positive number such that $k = \dfrac{CP'}{CP}$.
- Angle measures are preserved.



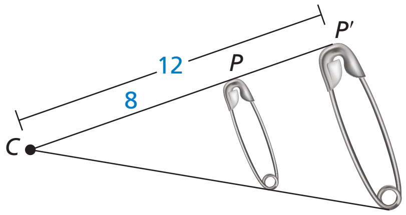
A dilation does not change any line that passes through the center of dilation. A dilation maps a line that does not pass through the center of dilation to a parallel line. In the figure above, $\overleftrightarrow{PR} \parallel \overleftrightarrow{P'R'}$, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$, and $\overleftrightarrow{QR} \parallel \overleftrightarrow{Q'R'}$.

When the scale factor $k > 1$, a dilation is an **enlargement**. When $0 < k < 1$, a dilation is a **reduction**.

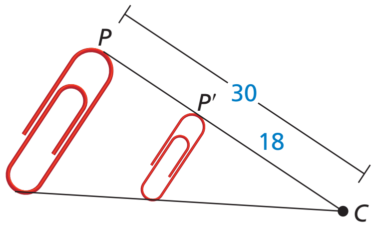
EXAMPLE 1 **Identifying Dilations**

Find the scale factor of the dilation. Then tell whether the dilation is a *reduction* or an *enlargement*.

a.



b.



SOLUTION

a. Because $\dfrac{CP'}{CP} = \dfrac{12}{8}$, the scale factor is $k = \dfrac{3}{2}$. So, the dilation is an enlargement.

b. Because $\dfrac{CP'}{CP} = \dfrac{18}{30}$, the scale factor is $k = \dfrac{3}{5}$. So, the dilation is a reduction.

READING

The scale factor of a dilation can be written as a fraction, decimal, or percent.

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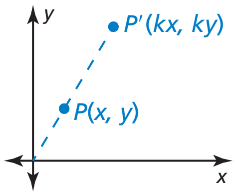
1. In a dilation, $CP' = 3$ and $CP = 12$. Find the scale factor. Then tell whether the dilation is a *reduction* or an *enlargement*.

Core Concept

Coordinate Rule for Dilations

If $P(x, y)$ is the preimage of a point, then its image after a dilation centered at the origin (0, 0) with scale factor k is the point $P'(kx, ky)$.



$(x, y) \rightarrow (kx, ky)$

READING DIAGRAMS

In this chapter, for all of the dilations in the coordinate plane, the center of dilation is the origin unless otherwise noted.

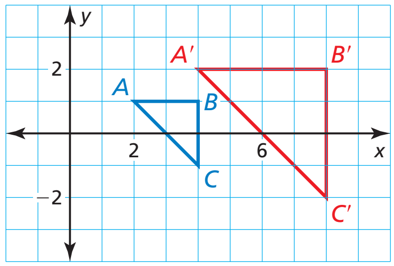
EXAMPLE 2 Dilating a Figure in the Coordinate Plane

Graph $\triangle ABC$ with vertices $A(2, 1)$, $B(4, 1)$, and $C(4, -1)$ and its image after a dilation with a scale factor of 2.

SOLUTION

Use the coordinate rule for a dilation with $k = 2$ to find the coordinates of the vertices of the image. Then graph $\triangle ABC$ and its image.

$$(x, y) \rightarrow (2x, 2y)$$
$$A(2, 1) \rightarrow A'(4, 2)$$
$$B(4, 1) \rightarrow B'(8, 2)$$
$$C(4, -1) \rightarrow C'(8, -2)$$



Notice the relationships between the lengths and slopes of the sides of the triangles in Example 2. Each side length of $\triangle A'B'C'$ is longer than its corresponding side by the scale factor. The corresponding sides are parallel because their slopes are the same.

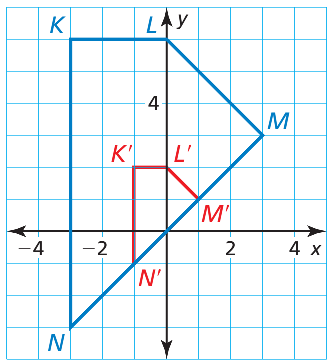
EXAMPLE 3 Dilating a Figure in the Coordinate Plane

Graph quadrilateral $KLMN$ with vertices $K(-3, 6)$, $L(0, 6)$, $M(3, 3)$, and $N(-3, -3)$ and its image after a dilation with a scale factor of $\frac{1}{3}$.

SOLUTION

Use the coordinate rule for a dilation with $k = \frac{1}{3}$ to find the coordinates of the vertices of the image. Then graph quadrilateral $KLMN$ and its image.

$$(x, y) \rightarrow \left(\tfrac{1}{3}x, \tfrac{1}{3}y\right)$$
$$K(-3, 6) \rightarrow K'(-1, 2)$$
$$L(0, 6) \rightarrow L'(0, 2)$$
$$M(3, 3) \rightarrow M'(1, 1)$$
$$N(-3, -3) \rightarrow N'(-1, -1)$$



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Graph $\triangle PQR$ and its image after a dilation with scale factor k.

2. $P(-2, -1)$, $Q(-1, 0)$, $R(0, -1)$; $k = 4$

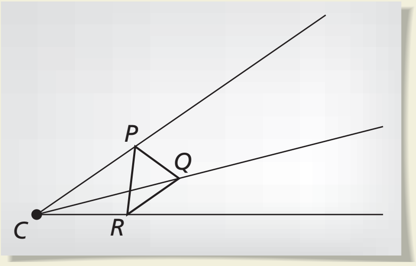
3. $P(5, -5)$, $Q(10, -5)$, $R(10, 5)$; $k = 0.4$

CONSTRUCTION Constructing a Dilation

Use a compass and straightedge to construct a dilation of $\triangle PQR$ with a scale factor of 2. Use a point C outside the triangle as the center of dilation.

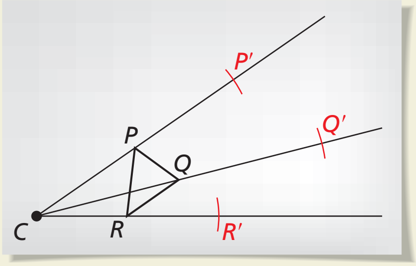
SOLUTION

Step 1



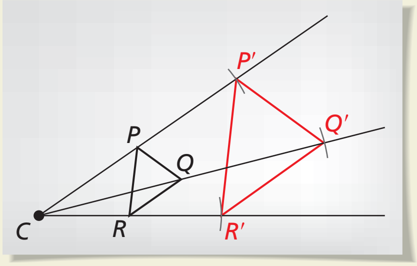
Draw a triangle Draw $\triangle PQR$ and choose the center of the dilation C outside the triangle. Draw rays from C through the vertices of the triangle.

Step 2

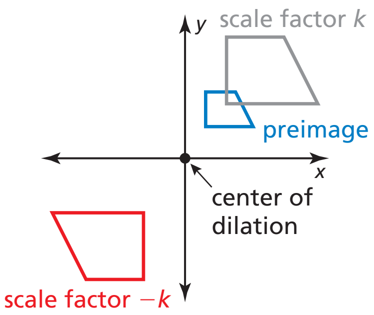


Use a compass Use a compass to locate P' on $\overrightarrow{CP}$ so that $CP' = 2(CP)$. Locate Q' and R' using the same method.

Step 3



Connect points Connect points P', Q', and R' to form $\triangle P'Q'R'$.



In the coordinate plane, you can have scale factors that are negative numbers. When this occurs, the figure rotates 180°. So, when $k > 0$, a dilation with a scale factor of $-k$ is the same as the composition of a dilation with a scale factor of k followed by a rotation of 180° about the center of dilation. Using the coordinate rules for a dilation and a rotation of 180°, you can think of the notation as

$$(x, y) \rightarrow (kx, ky) \rightarrow (-kx, -ky).$$

EXAMPLE 4 Using a Negative Scale Factor

Graph $\triangle FGH$ with vertices $F(-4, -2)$, $G(-2, 4)$, and $H(-2, -2)$ and its image after a dilation with a scale factor of $-\frac{1}{2}$.

SOLUTION

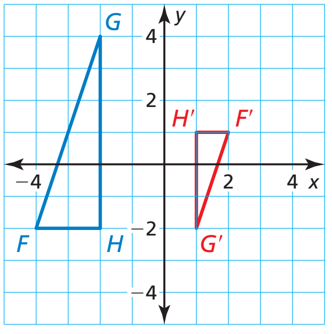
Use the coordinate rule for a dilation with $k = -\frac{1}{2}$ to find the coordinates of the vertices of the image. Then graph $\triangle FGH$ and its image.

$$(x, y) \rightarrow \left(-\tfrac{1}{2}x, -\tfrac{1}{2}y\right)$$

$F(-4, -2) \rightarrow F'(2, 1)$

$G(-2, 4) \rightarrow G'(1, -2)$

$H(-2, -2) \rightarrow H'(1, 1)$



Monitoring Progress

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4. Graph $\triangle PQR$ with vertices $P(1, 2)$, $Q(3, 1)$, and $R(1, -3)$ and its image after a dilation with a scale factor of -2.

5. Suppose a figure containing the origin is dilated. Explain why the corresponding point in the image of the figure is also the origin.

Solving Real-Life Problems

EXAMPLE 5 Finding a Scale Factor

You are making your own photo stickers. Your photo is 4 inches by 4 inches. The image on the stickers is 1.1 inches by 1.1 inches. What is the scale factor of this dilation?

READING

Scale factors are written so that the units in the numerator and denominator divide out.

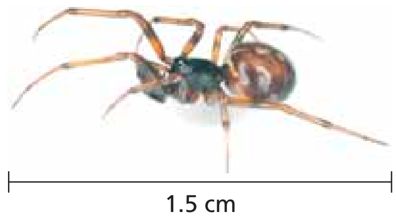
SOLUTION

The scale factor is the ratio of a side length of the sticker image to a side length of the original photo, or $\frac{1.1 \text{ in.}}{4 \text{ in.}}$.

▶ So, in simplest form, the scale factor is $\frac{11}{40}$.

EXAMPLE 6 Finding the Length of an Image

You are using a magnifying glass that shows the image of an object that is six times the object's actual size. Determine the length of the image of the spider seen through the magnifying glass.



SOLUTION

$$\frac{\text{image length}}{\text{actual length}} = k$$

$$\frac{x}{1.5} = 6$$

$$x = 9$$

▶ So, the image length through the magnifying glass is 9 centimeters.

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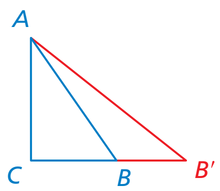
6. An optometrist dilates the pupils of a patient's eyes to get a better look at the back of the eyes. A pupil dilates from 4.5 millimeters to 8 millimeters. What is the scale factor of this dilation?

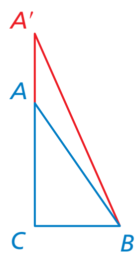
7. The image of a spider seen through the magnifying glass in Example 6 is shown at the left. Find the actual length of the spider.

When a transformation, such as a dilation, changes the shape or size of a figure, the transformation is *nonrigid*. In addition to dilations, there are many possible nonrigid transformations. Two examples are shown below. It is important to pay close attention to whether a nonrigid transformation preserves lengths and angle measures.

Horizontal Stretch



Vertical Stretch



4.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

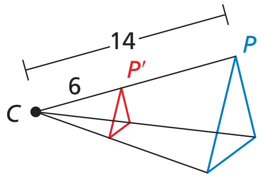
1. **COMPLETE THE SENTENCE** If $P(x, y)$ is the preimage of a point, then its image after a dilation centered at the origin $(0, 0)$ with scale factor k is the point ____________.

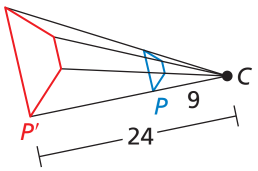
2. **WHICH ONE DOESN'T BELONG?** Which scale factor does *not* belong with the other three? Explain your reasoning.

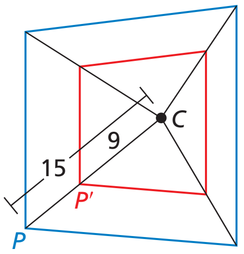
$\frac{5}{4}$	60%	115%	2

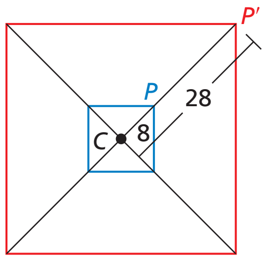
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the scale factor of the dilation. Then tell whether the dilation is a *reduction* or an *enlargement*. *(See Example 1.)*

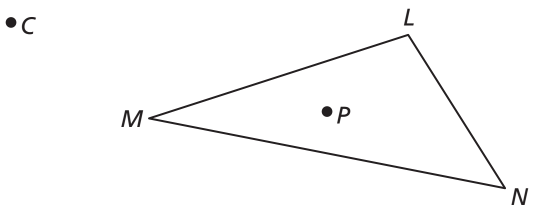
3. 

4. 

5. 

6. 

CONSTRUCTION **In Exercises 7–10, copy the diagram. Then use a compass and straightedge to construct a dilation of $\triangle LMN$ with the given center and scale factor k.**



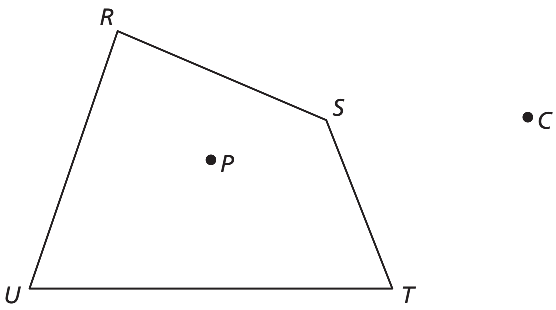
7. Center C, $k = 2$

8. Center P, $k = 3$

9. Center M, $k = \frac{1}{2}$

10. Center C, $k = 25\%$

CONSTRUCTION **In Exercises 11–14, copy the diagram. Then use a compass and straightedge to construct a dilation of quadrilateral $RSTU$ with the given center and scale factor k.**



11. Center C, $k = 3$

12. Center P, $k = 2$

13. Center R, $k = 0.25$

14. Center C, $k = 75\%$

In Exercises 15–18, graph the polygon and its image after a dilation with scale factor k. *(See Examples 2 and 3.)*

15. $X(6, -1)$, $Y(-2, -4)$, $Z(1, 2)$; $k = 3$

16. $A(0, 5)$, $B(-10, -5)$, $C(5, -5)$; $k = 120\%$

17. $T(9, -3)$, $U(6, 0)$, $V(3, 9)$, $W(0, 0)$; $k = \frac{2}{3}$

18. $J(4, 0)$, $K(-8, 4)$, $L(0, -4)$, $M(12, -8)$; $k = 0.25$

In Exercises 19–22, graph the polygon and its image after a dilation with scale factor k. *(See Example 4.)*

19. $B(-5, -10)$, $C(-10, 15)$, $D(0, 5)$; $k = -\frac{1}{5}$

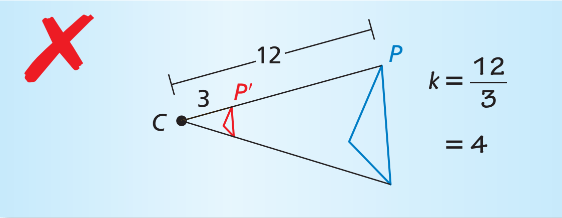
20. $L(0, 0)$, $M(-4, 1)$, $N(-3, -6)$; $k = -3$

21. $R(-7, -1)$, $S(2, 5)$, $T(-2, -3)$, $U(-3, -3)$; $k = -4$

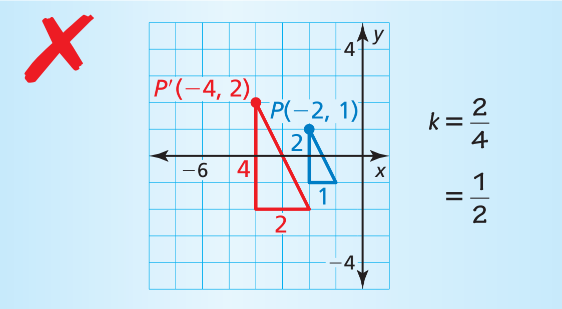
22. $W(8, -2)$, $X(6, 0)$, $Y(-6, 4)$, $Z(-2, 2)$; $k = -0.5$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in finding the scale factor of the dilation.**

23.

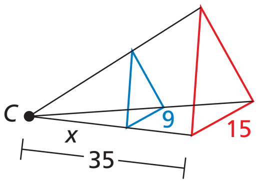


24.

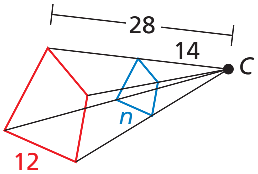


In Exercises 25–28, the red figure is the image of the blue figure after a dilation with center *C*. Find the scale factor of the dilation. Then find the value of the variable.

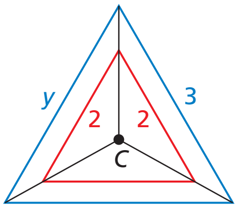
25.



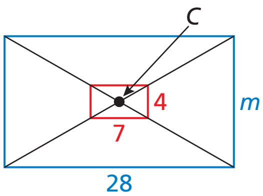
26.



27.



28.

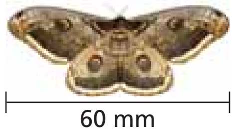


29. **FINDING A SCALE FACTOR** You receive wallet-sized photos of your school picture. The photo is 2.5 inches by 3.5 inches. You decide to dilate the photo to 5 inches by 7 inches at the store. What is the scale factor of this dilation? *(See Example 5.)*

30. **FINDING A SCALE FACTOR** Your visually impaired friend asked you to enlarge your notes from class so he can study. You took notes on 8.5-inch by 11-inch paper. The enlarged copy has a smaller side with a length of 10 inches. What is the scale factor of this dilation? *(See Example 5.)*

In Exercises 31–34, you are using a magnifying glass. Use the length of the insect and the magnification level to determine the length of the image seen through the magnifying glass. *(See Example 6.)*

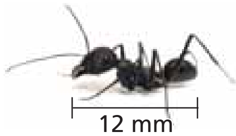
31. emperor moth
Magnification: 5×



32. ladybug
Magnification: 10×

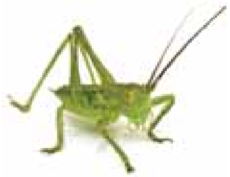
33. dragonfly
Magnification: 20×

34. carpenter ant
Magnification: 15×



35. **ANALYZING RELATIONSHIPS** Use the given actual and magnified lengths to determine which of the following insects were looked at using the same magnifying glass. Explain your reasoning.

grasshopper
Actual: 2 in.
Magnified: 15 in.



black beetle
Actual: 0.6 in.
Magnified: 4.2 in.

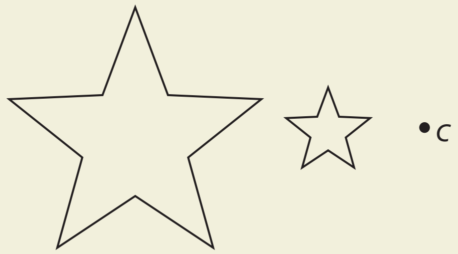
honeybee
Actual: $\frac{5}{8}$ in.
Magnified: $\frac{75}{16}$ in.

monarch butterfly
Actual: 3.9 in.
Magnified: 29.25 in.

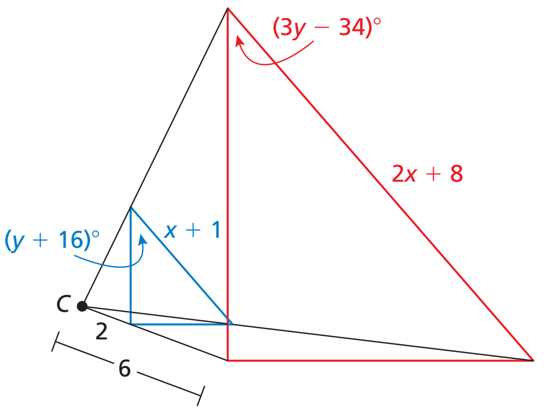
36. **THOUGHT PROVOKING** Draw $\triangle ABC$ and $\triangle A'B'C'$ so that $\triangle A'B'C'$ is a dilation of $\triangle ABC$. Find the center of dilation and explain how you found it.

37. **REASONING** Your friend prints a 4-inch by 6-inch photo for you from the school dance. All you have is an 8-inch by 10-inch frame. Can you dilate the photo to fit the frame? Explain your reasoning.

38. **HOW DO YOU SEE IT?** Point C is the center of dilation of the images. The scale factor is $\frac{1}{3}$. Which figure is the original figure? Which figure is the dilated figure? Explain your reasoning.



39. **MATHEMATICAL CONNECTIONS** The larger triangle is a dilation of the smaller triangle. Find the values of x and y.



40. **WRITING** Explain why a scale factor of 2 is the same as 200%.

In Exercises 41–44, determine whether the dilated figure or the original figure is closer to the center of dilation. Use the given location of the center of dilation and scale factor k.

41. Center of dilation: inside the figure; $k = 3$

42. Center of dilation: inside the figure; $k = \frac{1}{2}$

43. Center of dilation: outside the figure; $k = 120\%$

44. Center of dilation: outside the figure; $k = 0.1$

45. **ANALYZING RELATIONSHIPS** Dilate the line through $O(0, 0)$ and $A(1, 2)$ using a scale factor of 2.

a. What do you notice about the lengths of $\overline{O'A'}$ and $\overline{OA}$?

b. What do you notice about $\overleftrightarrow{O'A'}$ and $\overleftrightarrow{OA}$?

46. **ANALYZING RELATIONSHIPS** Dilate the line through $A(0, 1)$ and $B(1, 2)$ using a scale factor of $\frac{1}{2}$.

a. What do you notice about the lengths of $\overline{A'B'}$ and $\overline{AB}$?

b. What do you notice about $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{AB}$?

47. **ATTENDING TO PRECISION** You are making a blueprint of your house. You measure the lengths of the walls of your room to be 11 feet by 12 feet. When you draw your room on the blueprint, the lengths of the walls are 8.25 inches by 9 inches. What scale factor dilates your room to the blueprint?

48. **MAKING AN ARGUMENT** Your friend claims that dilating a figure by 1 is the same as dilating a figure by -1 because the original figure will not be enlarged or reduced. Is your friend correct? Explain your reasoning.

49. **USING STRUCTURE** Rectangle $WXYZ$ has vertices $W(-3, -1)$, $X(-3, 3)$, $Y(5, 3)$, and $Z(5, -1)$.

a. Find the perimeter and area of the rectangle.

b. Dilate the rectangle using a scale factor of 3. Find the perimeter and area of the dilated rectangle. Compare with the original rectangle. What do you notice?

c. Repeat part (b) using a scale factor of $\frac{1}{4}$.

d. Make a conjecture for how the perimeter and area change when a figure is dilated.

50. **REASONING** You put a reduction of a page on the original page. Explain why there is a point that is in the same place on both pages.

51. **REASONING** $\triangle ABC$ has vertices $A(4, 2)$, $B(4, 6)$, and $C(7, 2)$. Find the coordinates of the vertices of the image after a dilation with center $(4, 0)$ and a scale factor of 2.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

The vertices of $\triangle ABC$ are $A(2, -1)$, $B(0, 4)$, and $C(-3, 5)$. Find the coordinates of the vertices of the image after the translation. *(Section 4.1)*

52. $(x, y) \rightarrow (x, y - 4)$

53. $(x, y) \rightarrow (x - 1, y + 3)$

54. $(x, y) \rightarrow (x + 3, y - 1)$

55. $(x, y) \rightarrow (x - 2, y)$

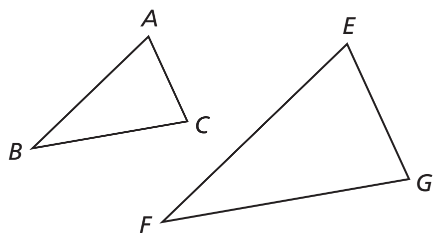
56. $(x, y) \rightarrow (x + 1, y - 2)$

57. $(x, y) \rightarrow (x - 3, y + 1)$

4.6 Similarity and Transformations

Essential Question When a figure is translated, reflected, rotated, or dilated in the plane, is the image always similar to the original figure?

Two figures are *similar figures* when they have the same shape but not necessarily the same size.



Similar Triangles

ATTENDING TO PRECISION

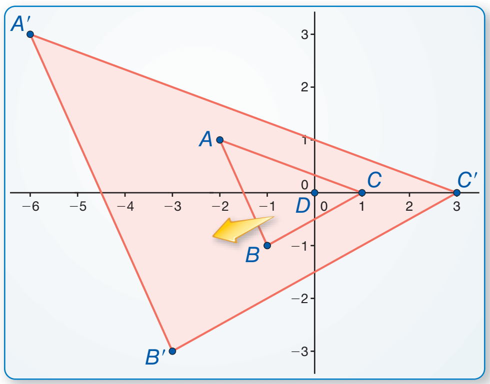
To be proficient in math, you need to use clear definitions in discussions with others and in your own reasoning.

EXPLORATION 1 Dilations and Similarity

Work with a partner.

a. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

b. Dilate the triangle using a scale factor of 3. Is the image similar to the original triangle? Justify your answer.



Sample

Points
$A(-2, 1)$
$B(-1, -1)$
$C(1, 0)$
$D(0, 0)$
Segments
$AB = 2.24$
$BC = 2.24$
$AC = 3.16$
Angles
$m\angle A = 45°$
$m\angle B = 90°$
$m\angle C = 45°$

EXPLORATION 2 Rigid Motions and Similarity

Work with a partner.

a. Use dynamic geometry software to draw any triangle.

b. Copy the triangle and translate it 3 units left and 4 units up. Is the image similar to the original triangle? Justify your answer.

c. Reflect the triangle in the y-axis. Is the image similar to the original triangle? Justify your answer.

d. Rotate the original triangle 90° counterclockwise about the origin. Is the image similar to the original triangle? Justify your answer.

Communicate Your Answer

3. When a figure is translated, reflected, rotated, or dilated in the plane, is the image always similar to the original figure? Explain your reasoning.

4. A figure undergoes a composition of transformations, which includes translations, reflections, rotations, and dilations. Is the image similar to the original figure? Explain your reasoning.

4.6 Lesson

Core Vocabulary

similarity transformation, *p. 216*
similar figures, *p. 216*

What You Will Learn

- Perform similarity transformations.
- Describe similarity transformations.
- Prove that figures are similar.

Performing Similarity Transformations

A dilation is a transformation that preserves shape but not size. So, a dilation is a nonrigid motion. A **similarity transformation** is a dilation or a composition of rigid motions and dilations. Two geometric figures are **similar figures** if and only if there is a similarity transformation that maps one of the figures onto the other. Similar figures have the same shape but not necessarily the same size.

Congruence transformations preserve length and angle measure. When the scale factor of the dilation(s) is not equal to 1 or -1, similarity transformations preserve angle measure only.

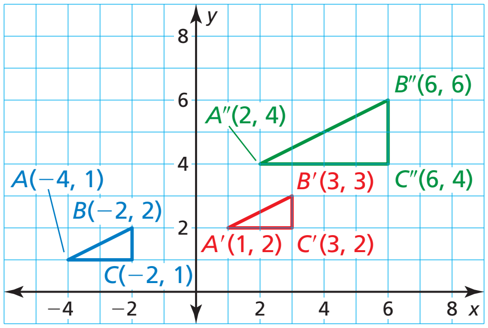
EXAMPLE 1 Performing a Similarity Transformation

Graph $\triangle ABC$ with vertices $A(-4, 1)$, $B(-2, 2)$, and $C(-2, 1)$ and its image after the similarity transformation.

Translation: $(x, y) \rightarrow (x + 5, y + 1)$
Dilation: $(x, y) \rightarrow (2x, 2y)$

SOLUTION

Step 1 Graph $\triangle ABC$.



Step 2 Translate $\triangle ABC$ 5 units right and 1 unit up. $\triangle A'B'C'$ has vertices $A'(1, 2)$, $B'(3, 3)$, and $C'(3, 2)$.

Step 3 Dilate $\triangle A'B'C'$ using a scale factor of 2. $\triangle A''B''C''$ has vertices $A''(2, 4)$, $B''(6, 6)$, and $C''(6, 4)$.

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1. Graph $\overline{CD}$ with endpoints $C(-2, 2)$ and $D(2, 2)$ and its image after the similarity transformation.

   Rotation: 90° about the origin

   Dilation: $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$

2. Graph $\triangle FGH$ with vertices $F(1, 2)$, $G(4, 4)$, and $H(2, 0)$ and its image after the similarity transformation.

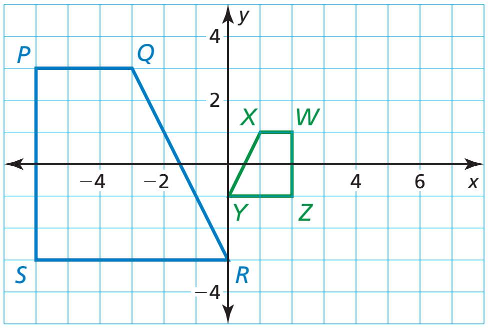
   Reflection: in the x-axis

   Dilation: $(x, y) \rightarrow (1.5x, 1.5y)$

Describing Similarity Transformations

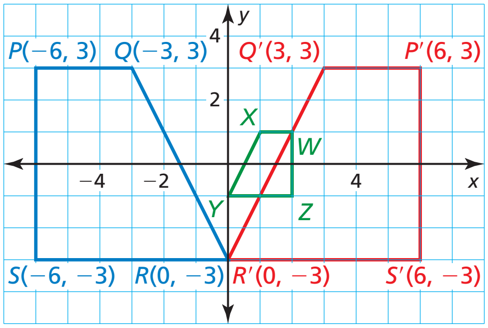
EXAMPLE 2 Describing a Similarity Transformation

Describe a similarity transformation that maps trapezoid *PQRS* to trapezoid *WXYZ*.



SOLUTION

$\overline{QR}$ falls from left to right, and $\overline{XY}$ rises from left to right. If you reflect trapezoid *PQRS* in the *y*-axis as shown, then the image, trapezoid $P'Q'R'S'$, will have the same orientation as trapezoid *WXYZ*.



Trapezoid *WXYZ* appears to be about one-third as large as trapezoid $P'Q'R'S'$. Dilate trapezoid $P'Q'R'S'$ using a scale factor of $\frac{1}{3}$.

$$(x, y) \rightarrow \left(\tfrac{1}{3}x, \tfrac{1}{3}y\right)$$

$$P'(6, 3) \rightarrow P''(2, 1)$$

$$Q'(3, 3) \rightarrow Q''(1, 1)$$

$$R'(0, -3) \rightarrow R''(0, -1)$$

$$S'(6, -3) \rightarrow S''(2, -1)$$

The vertices of trapezoid $P''Q''R''S''$ match the vertices of trapezoid *WXYZ*.

▶ So, a similarity transformation that maps trapezoid *PQRS* to trapezoid *WXYZ* is a reflection in the *y*-axis followed by a dilation with a scale factor of $\frac{1}{3}$.

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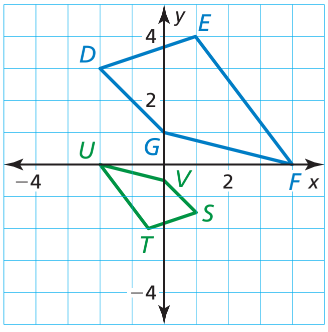
3. In Example 2, describe another similarity transformation that maps trapezoid *PQRS* to trapezoid *WXYZ*.

4. Describe a similarity transformation that maps quadrilateral *DEFG* to quadrilateral *STUV*.



Proving Figures Are Similar

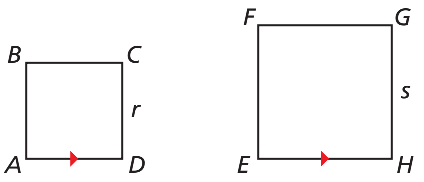
To prove that two figures are similar, you must prove that a similarity transformation maps one of the figures onto the other.

EXAMPLE 3 Proving That Two Squares Are Similar

Prove that square $ABCD$ is similar to square $EFGH$.

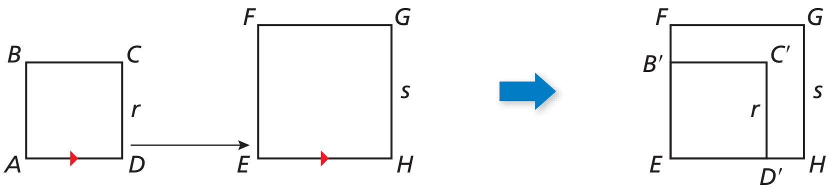
Given Square $ABCD$ with side length r, square $EFGH$ with side length s, $\overline{AD} \parallel \overline{EH}$

Prove Square $ABCD$ is similar to square $EFGH$.

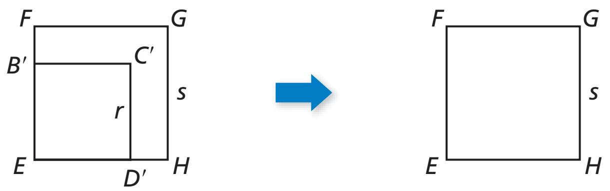


SOLUTION

Translate square $ABCD$ so that point A maps to point E. Because translations map segments to parallel segments and $\overline{AD} \parallel \overline{EH}$, the image of $\overline{AD}$ lies on $\overline{EH}$.



Because translations preserve length and angle measure, the image of $ABCD$, $EB'C'D'$, is a square with side length r. Because all the interior angles of a square are right angles, $\angle B'ED' \cong \angle FEH$. When $\overrightarrow{ED'}$ coincides with $\overrightarrow{EH}$, $\overrightarrow{EB'}$ coincides with $\overrightarrow{EF}$. So, $\overline{EB'}$ lies on $\overline{EF}$. Next, dilate square $EB'C'D'$ using center of dilation E. Choose the scale factor to be the ratio of the side lengths of $EFGH$ and $EB'C'D'$, which is $\frac{s}{r}$.

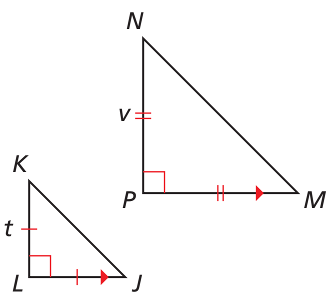


This dilation maps $\overline{ED'}$ to $\overline{EH}$ and $\overline{EB'}$ to $\overline{EF}$ because the images of $\overline{ED'}$ and $\overline{EB'}$ have side length $\frac{s}{r}(r) = s$ and the segments $\overline{ED'}$ and $\overline{EB'}$ lie on lines passing through the center of dilation. So, the dilation maps B' to F and D' to H. The image of C' lies $\frac{s}{r}(r) = s$ units to the right of the image of B' and $\frac{s}{r}(r) = s$ units above the image of D'. So, the image of C' is G.

▶ A similarity transformation maps square $ABCD$ to square $EFGH$. So, square $ABCD$ is similar to square $EFGH$.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*



5. Prove that $\triangle JKL$ is similar to $\triangle MNP$.

Given Right isosceles $\triangle JKL$ with leg length t, right isosceles $\triangle MNP$ with leg length v, $\overline{LJ} \parallel \overline{PM}$

Prove $\triangle JKL$ is similar to $\triangle MNP$.

4.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is the difference between *similar figures* and *congruent figures*?

2. **COMPLETE THE SENTENCE** A transformation that produces a similar figure, such as a dilation, is called a __________.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, graph $\triangle FGH$ with vertices $F(-2, 2)$, $G(-2, -4)$, and $H(-4, -4)$ and its image after the similarity transformation. *(See Example 1.)*

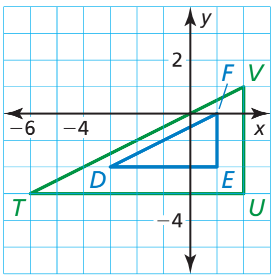
3. **Translation:** $(x, y) \rightarrow (x + 3, y + 1)$
   Dilation: $(x, y) \rightarrow (2x, 2y)$

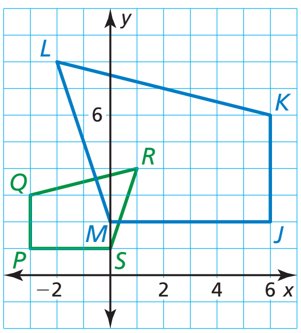
4. **Dilation:** $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$
   Reflection: in the y-axis

5. **Rotation:** 90° about the origin
   Dilation: $(x, y) \rightarrow (3x, 3y)$

6. **Dilation:** $(x, y) \rightarrow \left(\frac{3}{4}x, \frac{3}{4}y\right)$
   Reflection: in the x-axis

In Exercises 7 and 8, describe a similarity transformation that maps the blue preimage to the green image. *(See Example 2.)*

7. 

8. 

In Exercises 9–12, determine whether the polygons with the given vertices are similar. Use transformations to explain your reasoning.

9. $A(6, 0)$, $B(9, 6)$, $C(12, 6)$ and $D(0, 3)$, $E(1, 5)$, $F(2, 5)$

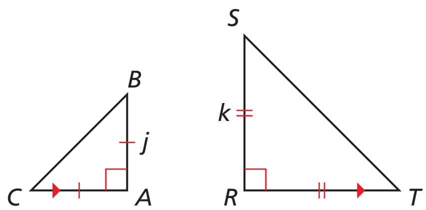
10. $Q(-1, 0)$, $R(-2, 2)$, $S(1, 3)$, $T(2, 1)$ and $W(0, 2)$, $X(4, 4)$, $Y(6, -2)$, $Z(2, -4)$

11. $G(-2, 3)$, $H(4, 3)$, $I(4, 0)$ and $J(1, 0)$, $K(6, -2)$, $L(1, -2)$

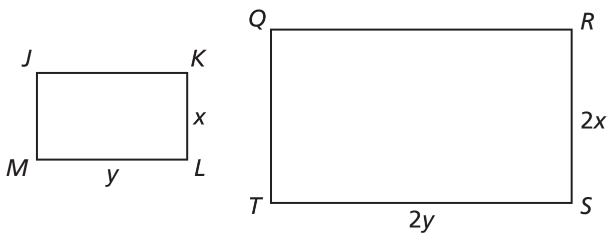
12. $D(-4, 3)$, $E(-2, 3)$, $F(-1, 1)$, $G(-4, 1)$ and $L(1, -1)$, $M(3, -1)$, $N(6, -3)$, $P(1, -3)$

In Exercises 13 and 14, prove that the figures are similar. *(See Example 3.)*

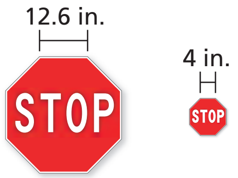
13. **Given** Right isosceles $\triangle ABC$ with leg length j, right isosceles $\triangle RST$ with leg length k, $\overline{CA} \parallel \overline{RT}$
    Prove $\triangle ABC$ is similar to $\triangle RST$.



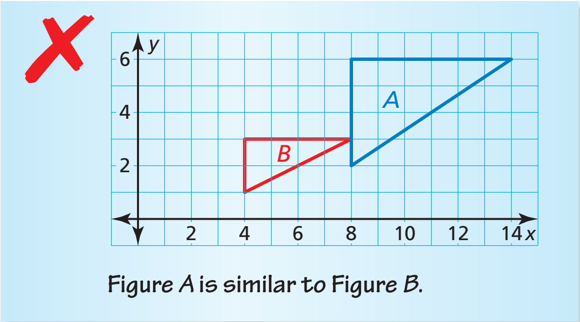
14. **Given** Rectangle $JKLM$ with side lengths x and y, rectangle $QRST$ with side lengths $2x$ and $2y$
    Prove Rectangle $JKLM$ is similar to rectangle $QRST$.



15. MODELING WITH MATHEMATICS Determine whether the regular-sized stop sign and the stop sign sticker are similar. Use transformations to explain your reasoning.

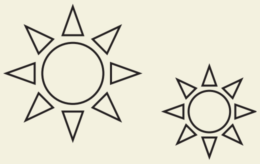


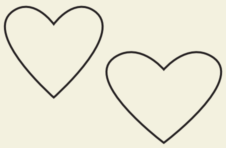
16. ERROR ANALYSIS Describe and correct the error in comparing the figures.



17. MAKING AN ARGUMENT A member of the homecoming decorating committee gives a printing company a banner that is 3 inches by 14 inches to enlarge. The committee member claims the banner she receives is distorted. Do you think the printing company distorted the image she gave it? Explain.

18. HOW DO YOU SEE IT? Determine whether each pair of figures is similar. Explain your reasoning.

a. 

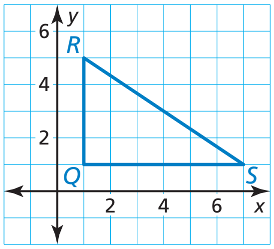
b. 

19. ANALYZING RELATIONSHIPS Graph a polygon in a coordinate plane. Use a similarity transformation involving a dilation (where k is a whole number) and a translation to graph a second polygon. Then describe a similarity transformation that maps the second polygon onto the first.

20. THOUGHT PROVOKING Is the composition of a rotation and a dilation commutative? (In other words, do you obtain the same image regardless of the order in which you perform the transformations?) Justify your answer.

21. MATHEMATICAL CONNECTIONS Quadrilateral $JKLM$ is mapped to quadrilateral $J'K'L'M'$ using the dilation $(x, y) \to \left(\frac{3}{2}x, \frac{3}{2}y\right)$. Then quadrilateral $J'K'L'M'$ is mapped to quadrilateral $J''K''L''M''$ using the translation $(x, y) \to (x + 3, y - 4)$. The vertices of quadrilateral $J'K'L'M'$ are $J'(-12, 0)$, $K'(-12, 18)$, $L'(-6, 18)$, and $M'(-6, 0)$. Find the coordinates of the vertices of quadrilateral $JKLM$ and quadrilateral $J''K''L''M''$. Are quadrilateral $JKLM$ and quadrilateral $J''K''L''M''$ similar? Explain.

22. REPEATED REASONING Use the diagram.



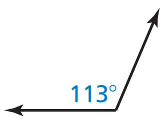
a. Connect the midpoints of the sides of $\triangle QRS$ to make another triangle. Is this triangle similar to $\triangle QRS$? Use transformations to support your answer.

b. Repeat part (a) for two other triangles. What conjecture can you make?

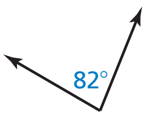
Maintaining Mathematical Proficiency

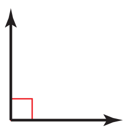
Reviewing what you learned in previous grades and lessons

Classify the angle as *acute*, *obtuse*, *right*, or *straight*. *(Section 1.5)*

23. 

24.

25. 

26. 

4.4–4.6 What Did You Learn?

Core Vocabulary

congruent figures, *p. 200*
congruence transformation, *p. 201*
dilation, *p. 208*
center of dilation, *p. 208*
scale factor, *p. 208*
enlargement, *p. 208*
reduction, *p. 208*
similarity transformation, *p. 216*
similar figures, *p. 216*

Core Concepts

Section 4.4

Identifying Congruent Figures, *p. 200*
Describing a Congruence Transformation, *p. 201*
Theorem 4.2 Reflections in Parallel Lines Theorem, *p. 202*
Theorem 4.3 Reflections in Intersecting Lines Theorem, *p. 203*

Section 4.5

Dilations and Scale Factor, *p. 208*
Coordinate Rule for Dilations, *p. 209*
Negative Scale Factors, *p. 210*

Section 4.6

Similarity Transformations, *p. 216*

Mathematical Practices

1. Revisit Exercise 31 on page 206. Try to recall the process you used to reach the solution. Did you have to change course at all? If so, how did you approach the situation?
2. Describe a real-life situation that can be modeled by Exercise 28 on page 213.

Performance Task

The Magic of Optics

Look at yourself in a shiny spoon. What happened to your reflection? Can you describe this mathematically? Now turn the spoon over and look at your reflection on the back of the spoon. What happened? Why?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

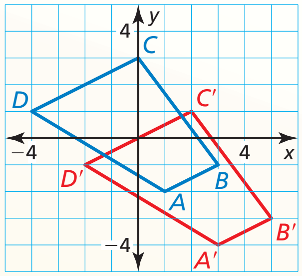
4 Chapter Review

Dynamic Solutions available at *BigIdeasMath.com*

4.1 Translations *(pp. 173–180)*

Graph quadrilateral *ABCD* with vertices $A(1, -2)$, $B(3, -1)$, $C(0, 3)$, and $D(-4, 1)$ and its image after the translation $(x, y) \rightarrow (x + 2, y - 2)$.

Graph quadrilateral *ABCD*. To find the coordinates of the vertices of the image, add 2 to the *x*-coordinates and subtract 2 from the *y*-coordinates of the vertices of the preimage. Then graph the image.



$$(x, y) \rightarrow (x + 2, y - 2)$$

$A(1, -2) \rightarrow A'(3, -4)$

$B(3, -1) \rightarrow B'(5, -3)$

$C(0, 3) \rightarrow C'(2, 1)$

$D(-4, 1) \rightarrow D'(-2, -1)$

Graph △*XYZ* with vertices $X(2, 3)$, $Y(-3, 2)$, and $Z(-4, -3)$ and its image after the translation.

1. $(x, y) \rightarrow (x, y + 2)$

2. $(x, y) \rightarrow (x - 3, y)$

3. $(x, y) \rightarrow (x + 3, y - 1)$

4. $(x, y) \rightarrow (x + 4, y + 1)$

Graph △*PQR* with vertices $P(0, -4)$, $Q(1, 3)$, and $R(2, -5)$ and its image after the composition.

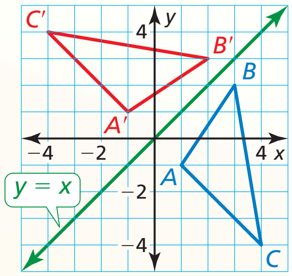
5. **Translation:** $(x, y) \rightarrow (x + 1, y + 2)$
Translation: $(x, y) \rightarrow (x - 4, y + 1)$

6. **Translation:** $(x, y) \rightarrow (x, y + 3)$
Translation: $(x, y) \rightarrow (x - 1, y + 1)$

4.2 Reflections *(pp. 181–188)*

Graph △*ABC* with vertices $A(1, -1)$, $B(3, 2)$, and $C(4, -4)$ and its image after a reflection in the line $y = x$.

Graph △*ABC* and the line $y = x$. Then use the coordinate rule for reflecting in the line $y = x$ to find the coordinates of the vertices of the image.



$$(a, b) \rightarrow (b, a)$$

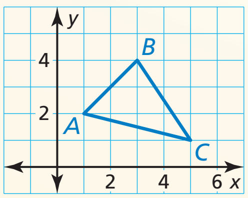
$A(1, -1) \rightarrow A'(-1, 1)$

$B(3, 2) \rightarrow B'(2, 3)$

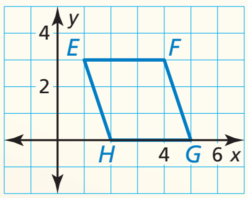
$C(4, -4) \rightarrow C'(-4, 4)$

Graph the polygon and its image after a reflection in the given line.

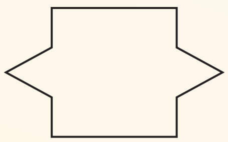
7. $x = 4$



8. $y = 3$



9. How many lines of symmetry does the figure have?



4.3 Rotations (pp. 189–196)

Graph $\triangle LMN$ with vertices $L(1, -1)$, $M(2, 3)$, and $N(4, 0)$ and its image after a 270° rotation about the origin.

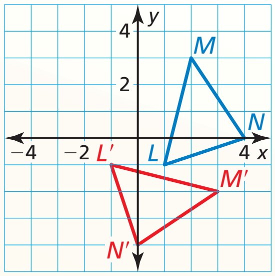
Use the coordinate rule for a 270° rotation to find the coordinates of the vertices of the image. Then graph $\triangle LMN$ and its image.

$$(a, b) \rightarrow (b, -a)$$

$L(1, -1) \rightarrow L'(-1, -1)$

$M(2, 3) \rightarrow M'(3, -2)$

$N(4, 0) \rightarrow N'(0, -4)$



Graph the polygon with the given vertices and its image after a rotation of the given number of degrees about the origin.

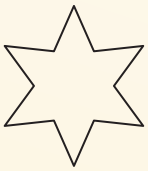
10. $A(-3, -1)$, $B(2, 2)$, $C(3, -3)$; 90°

11. $W(-2, -1)$, $X(-1, 3)$, $Y(3, 3)$, $Z(3, -3)$; 180°

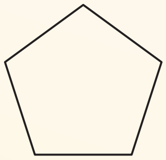
12. Graph $\overline{XY}$ with endpoints $X(5, -2)$ and $Y(3, -3)$ and its image after a reflection in the x-axis and then a rotation of 270° about the origin.

Determine whether the figure has rotational symmetry. If so, describe any rotations that map the figure onto itself.

13.

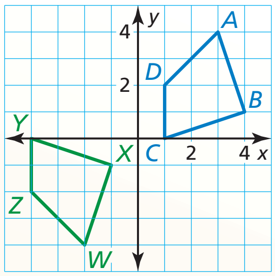


14.

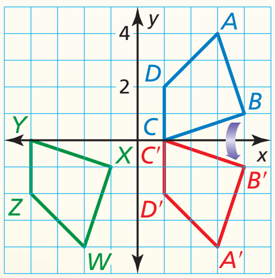


4.4 Congruence and Transformations (pp. 199–206)

Describe a congruence transformation that maps quadrilateral *ABCD* to quadrilateral *WXYZ*, as shown at the right.



$\overline{AB}$ falls from left to right, and $\overline{WX}$ rises from left to right. If you reflect quadrilateral *ABCD* in the x-axis as shown at the bottom right, then the image, quadrilateral $A'B'C'D'$, will have the same orientation as quadrilateral *WXYZ*. Then you can map quadrilateral $A'B'C'D'$ to quadrilateral *WXYZ* using a translation of 5 units left.



▶ So, a congruence transformation that maps quadrilateral *ABCD* to quadrilateral *WXYZ* is a reflection in the x-axis followed by a translation of 5 units left.

Describe a congruence transformation that maps $\triangle DEF$ to $\triangle JKL$.

15. $D(2, -1)$, $E(4, 1)$, $F(1, 2)$ and $J(-2, -4)$, $K(-4, -2)$, $L(-1, -1)$

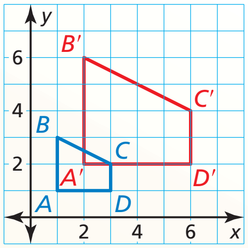
16. $D(-3, -4)$, $E(-5, -1)$, $F(-1, 1)$ and $J(1, 4)$, $K(-1, 1)$, $L(3, -1)$

17. Which transformation is the same as reflecting an object in two parallel lines? in two intersecting lines?

4.5 Dilations (pp. 207–214)

Graph trapezoid *ABCD* with vertices *A*(1, 1), *B*(1, 3), *C*(3, 2), and *D*(3, 1) and its image after a dilation with a scale factor of 2.

Use the coordinate rule for a dilation with $k = 2$ to find the coordinates of the vertices of the image. Then graph trapezoid *ABCD* and its image.



$$(x, y) \rightarrow (2x, 2y)$$

$A(1, 1) \rightarrow A'(2, 2)$

$B(1, 3) \rightarrow B'(2, 6)$

$C(3, 2) \rightarrow C'(6, 4)$

$D(3, 1) \rightarrow D'(6, 2)$

Graph the triangle and its image after a dilation with scale factor *k*.

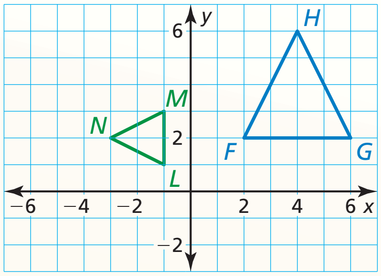
18. $P(2, 2), Q(4, 4), R(8, 2); k = \frac{1}{2}$

19. $X(-3, 2), Y(2, 3), Z(1, -1); k = -3$

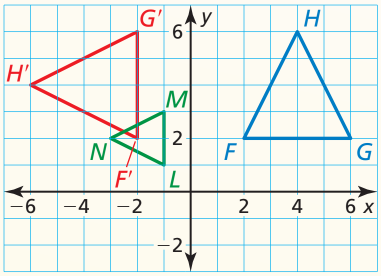
20. You are using a magnifying glass that shows the image of an object that is eight times the object's actual size. The image length is 15.2 centimeters. Find the actual length of the object.

4.6 Similarity and Transformations (pp. 215–220)

Describe a similarity transformation that maps $\triangle FGH$ to $\triangle LMN$, as shown at the right.



$\overline{FG}$ is horizontal, and $\overline{LM}$ is vertical. If you rotate $\triangle FGH$ 90° about the origin as shown at the bottom right, then the image, $\triangle F'G'H'$, will have the same orientation as $\triangle LMN$. $\triangle LMN$ appears to be half as large as $\triangle F'G'H'$. Dilate $\triangle F'G'H'$ using a scale factor of $\frac{1}{2}$.



$$(x, y) \rightarrow \left(\tfrac{1}{2}x, \tfrac{1}{2}y\right)$$

$F'(-2, 2) \rightarrow F''(-1, 1)$

$G'(-2, 6) \rightarrow G''(-1, 3)$

$H'(-6, 4) \rightarrow H''(-3, 2)$

The vertices of $\triangle F''G''H''$ match the vertices of $\triangle LMN$.

► So, a similarity transformation that maps $\triangle FGH$ to $\triangle LMN$ is a rotation of 90° about the origin followed by a dilation with a scale factor of $\frac{1}{2}$.

Describe a similarity transformation that maps $\triangle ABC$ to $\triangle RST$.

21. $A(1, 0), B(-2, -1), C(-1, -2)$ and $R(-3, 0), S(6, -3), T(3, -6)$

22. $A(6, 4), B(-2, 0), C(-4, 2)$ and $R(2, 3), S(0, -1), T(1, -2)$

23. $A(3, -2), B(0, 4), C(-1, -3)$ and $R(-4, -6), S(8, 0), T(-6, 2)$

4 Chapter Test

Graph △*RST* with vertices *R*(−4, 1), *S*(−2, 2), and *T*(3, −2) and its image after the translation.

1. $(x, y) \rightarrow (x - 4, y + 1)$

2. $(x, y) \rightarrow (x + 2, y - 2)$

Graph the polygon with the given vertices and its image after a rotation of the given number of degrees about the origin.

3. $D(-1, -1)$, $E(-3, 2)$, $F(1, 4)$; 270°

4. $J(-1, 1)$, $K(3, 3)$, $L(4, -3)$, $M(0, -2)$; 90°

Determine whether the polygons with the given vertices are congruent or similar. Use transformations to explain your reasoning.

5. $Q(2, 4)$, $R(5, 4)$, $S(6, 2)$, $T(1, 2)$ and $W(6, -12)$, $X(15, -12)$, $Y(18, -6)$, $Z(3, -6)$

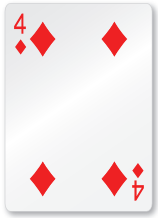
6. $A(-6, 6)$, $B(-6, 2)$, $C(-2, -4)$ and $D(9, 7)$, $E(5, 7)$, $F(-1, 3)$

Determine whether the object has line symmetry and whether it has rotational symmetry. Identify all lines of symmetry and angles of rotation that map the figure onto itself.

7.

8.

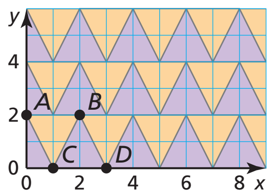
9.

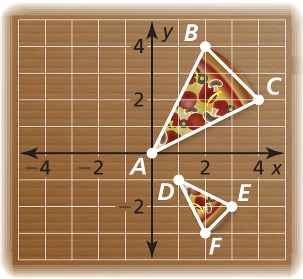


10. Draw a diagram using a coordinate plane, two parallel lines, and a parallelogram that demonstrates the Reflections in Parallel Lines Theorem (Theorem 4.2).

11. A rectangle with vertices $W(-2, 4)$, $X(2, 4)$, $Y(2, 2)$, and $Z(-2, 2)$ is reflected in the y-axis. Your friend says that the image, rectangle $W'X'Y'Z'$, is exactly the same as the preimage. Is your friend correct? Explain your reasoning.

12. Write a composition of transformations that maps △*ABC* onto △*CDB* in the tessellation shown. Is the composition a congruence transformation? Explain your reasoning.





13. There is one slice of a large pizza and one slice of a small pizza in the box.

a. Describe a similarity transformation that maps pizza slice *ABC* to pizza slice *DEF*.

b. What is one possible scale factor for a medium slice of pizza? Explain your reasoning. (Use a dilation on the large slice of pizza.)

14. The original photograph shown is 4 inches by 6 inches.

a. What transformations can you use to produce the new photograph?

b. You dilate the original photograph by a scale factor of $\frac{1}{2}$. What are the dimensions of the new photograph?

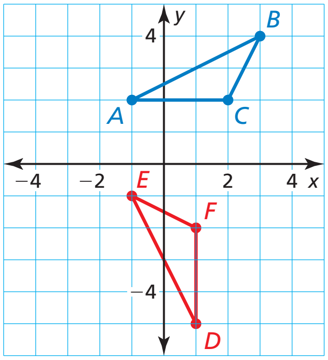
c. You have a frame that holds photos that are 8.5 inches by 11 inches. Can you dilate the original photograph to fit the frame? Explain your reasoning.

new

original

4 Cumulative Assessment

1. Which composition of transformations maps $\triangle ABC$ to $\triangle DEF$?



Ⓐ **Rotation:** 90° counterclockwise about the origin
Translation: $(x, y) \rightarrow (x + 4, y - 3)$

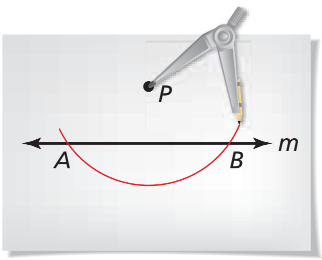
Ⓑ **Translation:** $(x, y) \rightarrow (x - 4, y - 3)$
Rotation: 90° counterclockwise about the origin

Ⓒ **Translation:** $(x, y) \rightarrow (x + 4, y - 3)$
Rotation: 90° counterclockwise about the origin

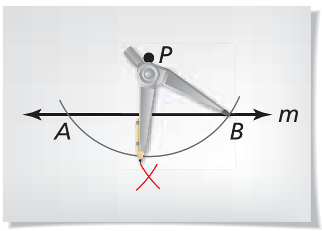
Ⓓ **Rotation:** 90° counterclockwise about the origin
Translation: $(x, y) \rightarrow (x - 4, y - 3)$

2. Use the diagrams to describe the steps you would take to construct a line perpendicular to line m through point P, which is not on line m.

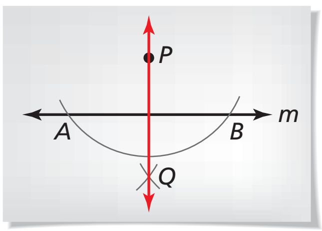
Step 1



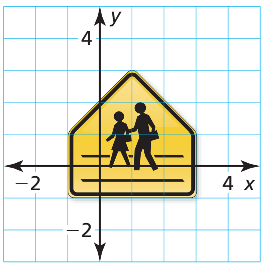
Step 2



Step 3

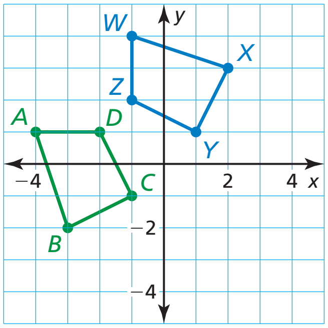


3. Your friend claims that she can find the perimeter of the school crossing sign without using the Distance Formula. Do you support your friend's claim? Explain your reasoning.



4. Graph the directed line segment ST with endpoints $S(-3, -2)$ and $T(4, 5)$. Then find the coordinates of point P along the directed line segment ST so that the ratio of SP to PT is 3 to 4.

5. The graph shows quadrilateral $WXYZ$ and quadrilateral $ABCD$.



a. Write a composition of transformations that maps quadrilateral $WXYZ$ to quadrilateral $ABCD$.

b. Are the quadrilaterals congruent? Explain your reasoning.

6. Which equation represents the line passing through the point $(-6, 3)$ that is parallel to the line $y = -\frac{1}{3}x - 5$?

Ⓐ $y = 3x + 21$

Ⓑ $y = -\frac{1}{3}x - 5$

Ⓒ $y = 3x - 15$

Ⓓ $y = -\frac{1}{3}x + 1$

7. Which scale factor(s) would create a dilation of $\overline{AB}$ that is shorter than $\overline{AB}$? Select all that apply.

$\frac{1}{3}$ $\frac{1}{2}$ $\frac{3}{4}$ 1 $\frac{3}{2}$ 2 3 $\frac{7}{2}$

8. List one possible set of coordinates of the vertices of quadrilateral $ABCD$ for each description.

a. A reflection in the y-axis maps quadrilateral $ABCD$ onto itself.

b. A reflection in the x-axis maps quadrilateral $ABCD$ onto itself.

c. A rotation of 90° about the origin maps quadrilateral $ABCD$ onto itself.

d. A rotation of 180° about the origin maps quadrilateral $ABCD$ onto itself.