Chapter 7: Polynomial Equations and Factoring

- 7.1 Adding and Subtracting Polynomials
- 7.2 Multiplying Polynomials
- 7.3 Special Products of Polynomials
- 7.4 Solving Polynomial Equations in Factored Form
- 7.5 Factoring $x^2 + bx + c$
- 7.6 Factoring $ax^2 + bx + c$
- 7.7 Factoring Special Products
- 7.8 Factoring Polynomials Completely

SEE the Big Idea

- Height of a Falling Object (p. 400)
- Game Reserve (p. 394)
- Photo Cropping (p. 390)
- Framing a Photo (p. 370)
- Gateway Arch (p. 382)
Maintaining Mathematical Proficiency

Simplifying Algebraic Expressions

Example 1  Simplify $6x + 5 - 3x - 4$.

\[
\begin{align*}
6x + 5 - 3x - 4 &= 6x - 3x + 5 - 4 & \text{Commutative Property of Addition} \\
&= (6 - 3)x + 5 - 4 & \text{Distributive Property} \\
&= 3x + 1 & \text{Simplify.}
\end{align*}
\]

Example 2  Simplify $-8(y - 3) + 2y$.

\[
\begin{align*}
-8(y - 3) + 2y &= -8(y) - (-8)(3) + 2y & \text{Distributive Property} \\
&= -8y + 24 + 2y & \text{Multiply.} \\
&= -8y + 2y + 24 & \text{Commutative Property of Addition} \\
&= (-8 + 2)y + 24 & \text{Distributive Property} \\
&= -6y + 24 & \text{Simplify.}
\end{align*}
\]

Simplify the expression.

1. $3x - 7 + 2x$
2. $4r + 6 - 9r - 1$
3. $-5t + 3 - t - 4 + 8t$
4. $3(s - 1) + 5$
5. $2m - 7(3 - m)$
6. $4(h + 6) - (h - 2)$

Finding the Greatest Common Factor

Example 3  Find the greatest common factor (GCF) of 42 and 70.

To find the GCF of two numbers, first write the prime factorization of each number. Then find the product of the common prime factors.

\[
\begin{align*}
42 &= 2 \cdot 3 \cdot 7 \\
70 &= 2 \cdot 5 \cdot 7
\end{align*}
\]

The GCF of 42 and 70 is $2 \cdot 7 = 14$.

Find the greatest common factor.

7. 20, 36
8. 42, 63
9. 54, 81
10. 72, 84
11. 28, 64
12. 30, 77
13. **ABSTRACT REASONING** Is it possible for two integers to have no common factors? Explain your reasoning.
Mathematically proficient students consider concrete models when solving a mathematics problem.

Using Models

**Core Concept**

**Using Algebra Tiles**

When solving a problem, it can be helpful to use a model. For instance, you can use algebra tiles to model algebraic expressions and operations with algebraic expressions.

```
1  -1  x  -x  x²  -x²
```

**EXAMPLE 1** Writing Expressions Modeled by Algebra Tiles

Write the algebraic expression modeled by the algebra tiles.

a. The algebraic expression is $x^2$.

b. The algebraic expression is $3x + 4$.

c. The algebraic expression is $x^2 - x + 2$.

**Monitoring Progress**

Write the algebraic expression modeled by the algebra tiles.

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

9.
7.1 Adding and Subtracting Polynomials

**Essential Question** How can you add and subtract polynomials?

**EXPLORATION 1** Adding Polynomials

Work with a partner. Write the expression modeled by the algebra tiles in each step.

**Step 1**

\[ 3x + 2 + (x - 5) \]

**Step 2**

**Step 3**

**Step 4**

---

**EXPLORATION 2** Subtracting Polynomials

Work with a partner. Write the expression modeled by the algebra tiles in each step.

**Step 1**

\[ (x^2 + 2x + 2) - (x - 1) \]

**Step 2**

**Step 3**

**Step 4**

**Step 5**

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**Communicate Your Answer**

3. How can you add and subtract polynomials?

4. Use your methods in Question 3 to find each sum or difference.

   a. \( (x^2 + 2x - 1) + (2x^2 - 2x + 1) \)
   
   b. \( (4x + 3) + (x - 2) \)
   
   c. \( (x^2 + 2) - (3x^2 + 2x + 5) \)
   
   d. \( (2x - 3x) - (x^2 - 2x + 4) \)
7.1 Lesson

What You Will Learn

- Find the degrees of monomials.
- Classify polynomials.
- Add and subtract polynomials.
- Solve real-life problems.

Finding the Degrees of Monomials

A **monomial** is a number, a variable, or the product of a number and one or more variables with whole number exponents.

The **degree of a monomial** is the sum of the exponents of the variables in the monomial. The degree of a nonzero constant term is 0. The constant 0 does not have a degree.

<table>
<thead>
<tr>
<th>Monomial</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>3x</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{2} \text{ab}^2 )</td>
<td>1 + 2 = 3</td>
</tr>
<tr>
<td>(-1.8m^5 )</td>
<td>5</td>
</tr>
<tr>
<td>( 5 + x )</td>
<td>A sum is not a monomial.</td>
</tr>
<tr>
<td>( \frac{2}{n} )</td>
<td>A monomial cannot have a variable in the denominator.</td>
</tr>
<tr>
<td>( 4^a )</td>
<td>A monomial cannot have a variable exponent.</td>
</tr>
<tr>
<td>( x^{-1} )</td>
<td>The variable must have a whole number exponent.</td>
</tr>
</tbody>
</table>

**EXAMPLE 1** Finding the Degrees of Monomials

Find the degree of each monomial.

a. \( 5x^2 \)   
   
   
   b. \( -\frac{1}{2}xy^3 \)   
   
   
   c. \( 8x^3y^3 \)   
   
   
   d. \( -3 \)

**SOLUTION**

a. The exponent of \( x \) is 2.
   
   So, the degree of the monomial is 2.

b. The exponent of \( x \) is 1, and the exponent of \( y \) is 3.
   
   So, the degree of the monomial is 1 + 3, or 4.

c. The exponent of \( x \) is 3, and the exponent of \( y \) is 3.
   
   So, the degree of the monomial is 3 + 3, or 6.

d. You can rewrite \(-3\) as \(-3x^0\).
   
   So, the degree of the monomial is 0.

**Monitoring Progress**

Find the degree of the monomial.

1. \( -3x^4 \)   
2. \( 7e^3d^2 \)   
3. \( \frac{5}{3}y \)   
4. \(-20.5\)
Classifying Polynomials

**Core Concept**

**Polynomials**

A polynomial is a monomial or a sum of monomials. Each monomial is called a term of the polynomial. A polynomial with two terms is a binomial. A polynomial with three terms is a trinomial.

<table>
<thead>
<tr>
<th>Binomial</th>
<th>Trinomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5x + 2 )</td>
<td>( x^2 + 5x + 2 )</td>
</tr>
</tbody>
</table>

The degree of a polynomial is the greatest degree of its terms. A polynomial in one variable is in standard form when the exponents of the terms decrease from left to right. When you write a polynomial in standard form, the coefficient of the first term is the leading coefficient.

**Example 2** Writing a Polynomial in Standard Form

Write \( 15x - x^3 + 3 \) in standard form. Identify the degree and leading coefficient of the polynomial.

**Solution**

Consider the degree of each term of the polynomial.

- Degree is 3.
- Degree is 1.
- Degree is 0.

You can write the polynomial in standard form as \(-x^3 + 15x + 3\). The greatest degree is 3, so the degree of the polynomial is 3, and the leading coefficient is \(-1\).

**Example 3** Classifying Polynomials

Write each polynomial in standard form. Identify the degree and classify each polynomial by the number of terms.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Standard Form</th>
<th>Degree</th>
<th>Type of Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (-3z^4)</td>
<td>(-3z^4)</td>
<td>4</td>
<td>monomial</td>
</tr>
<tr>
<td>b. (4 + 5x^2 - x)</td>
<td>(5x^2 - x + 4)</td>
<td>2</td>
<td>trinomial</td>
</tr>
<tr>
<td>c. (8q + q^5)</td>
<td>(q^5 + 8q)</td>
<td>5</td>
<td>binomial</td>
</tr>
</tbody>
</table>

**Monitoring Progress**

Write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms.

5. \(4 - 9z\)  
6. \(t^2 - t^3 - 10t\)  
7. \(2.8x + x^3\)
Adding and Subtracting Polynomials

A set of numbers is **closed** under an operation when the operation performed on any two numbers in the set results in a number that is also in the set. For example, the set of integers is closed under addition, subtraction, and multiplication. This means that if \(a\) and \(b\) are two integers, then \(a + b\), \(a - b\), and \(ab\) are also integers.

The set of polynomials is closed under addition and subtraction. So, the sum or difference of any two polynomials is also a polynomial.

To add polynomials, add like terms. You can use a vertical or a horizontal format.

### Example 4 Adding Polynomials

Find the sum.

a. \((2x^3 - 5x^2 + x) + (2x^2 + x^3 - 1)\)  
   b. \((3x^2 + x - 6) + (x^2 + 4x + 10)\)

**SOLUTION**

a. **Vertical format:** Align like terms vertically and add.

\[
\begin{align*}
2x^3 - 5x^2 + x \\
+ x^3 + 2x^2 - 1 \\
\hline
3x^3 - 3x^2 + x - 1
\end{align*}
\]

\[
\text{The sum is } 3x^3 - 3x^2 + x - 1.
\]

b. **Horizontal format:** Group like terms and simplify.

\[
(3x^2 + x - 6) + (x^2 + 4x + 10) = (3x^2 + x^2) + (x + 4x) + (-6 + 10)
\]

\[= 4x^2 + 5x + 4\]

\[
\text{The sum is } 4x^2 + 5x + 4.
\]

To subtract a polynomial, add its opposite. To find the opposite of a polynomial, multiply each of its terms by \(-1\).

### Example 5 Subtracting Polynomials

Find the difference.

a. \((4n^2 + 5) - (-2n^2 + 2n - 4)\)  
   b. \((4x^2 - 3x + 5) - (3x^2 - x - 8)\)

**SOLUTION**

a. **Vertical format:** Align like terms vertically and subtract.

\[
\begin{align*}
4n^2 & \quad + 5 \\
- (-2n^2 + 2n - 4) & \quad \rightarrow \\
\hline
6n^2 & \quad - 2n + 9
\end{align*}
\]

\[
\text{The difference is } 6n^2 - 2n + 9.
\]

b. **Horizontal format:** Group like terms and simplify.

\[
(4x^2 - 3x + 5) - (3x^2 - x - 8) = 4x^2 - 3x + 5 - 3x^2 + x + 8
\]

\[= (4x^2 - 3x^2) + (-3x + x) + (5 + 8)
\]

\[= x^2 - 2x + 13\]

\[
\text{The difference is } x^2 - 2x + 13.
\]
Monitoring Progress

Find the sum or difference.

8. \((b - 10) + (4b - 3)\)  
9. \((x^2 - x - 2) + (7x^2 - x)\)  
10. \((p^2 + p + 3) - (-4p^2 - p + 3)\)  
11. \((-k + 5) - (3k^2 - 6)\)

Solving Real-Life Problems

EXAMPLE 6  Solving a Real-Life Problem

A penny is thrown straight down from a height of 200 feet. At the same time, a paintbrush is dropped from a height of 100 feet. The polynomials represent the heights (in feet) of the objects after \(t\) seconds.

\[
\begin{align*}
\text{Penny:} & \quad -16t^2 - 40t + 200 \\
\text{Paintbrush:} & \quad -16t^2 + 100
\end{align*}
\]

a. Write a polynomial that represents the distance between the penny and the paintbrush after \(t\) seconds.

b. Interpret the coefficients of the polynomial in part (a).

SOLUTION

a. To find the distance between the objects after \(t\) seconds, subtract the polynomials.

\[
\begin{align*}
\text{Distance} & = \text{Penny} - \text{Paintbrush} \\
& = (-16t^2 - 40t + 200) - (-16t^2 + 100) \\
& = -40t + 100
\end{align*}
\]

The polynomial \(-40t + 100\) represents the distance between the objects after \(t\) seconds.

b. When \(t = 0\), the distance between the objects is \(-40(0) + 100 = 100\) feet. So, the constant term 100 represents the distance between the penny and the paintbrush when both objects begin to fall.

As the value of \(t\) increases by 1, the value of \(-40t + 100\) decreases by 40. This means that the objects become 40 feet closer to each other each second. So, \(-40\) represents the amount that the distance between the objects changes each second.

Monitoring Progress

12. WHAT IF? The polynomial \(-16t^2 - 25t + 200\) represents the height of the penny after \(t\) seconds.

a. Write a polynomial that represents the distance between the penny and the paintbrush after \(t\) seconds.

b. Interpret the coefficients of the polynomial in part (a).
19. In Exercises 13–20, write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms. (See Examples 2 and 3.)

19. \( \pi r^2 - \frac{4}{3} \pi r^3 + 2r^5 \)

20. \( \sqrt{7n^4} \)

Vocabulary and Core Concept Check

1. **VOCABULARY** When is a polynomial in one variable in standard form?

2. **OPEN-ENDED** Write a trinomial in one variable of degree 5 in standard form.

3. **VOCABULARY** How can you determine whether a set of numbers is closed under an operation?

4. **WHICH ONE DOESN’T BELONG?** Which expression does not belong with the other three? Explain your reasoning.

\[ a^3 + 4a \quad x^2 - 8x \quad b - 2^{-1} \quad \frac{\pi}{3} + 6y^8z \]

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, find the degree of the monomial. (See Example 1.)

5. \( 4g \)

6. \( 23x^4 \)

7. \( -1.75k^2 \)

8. \( -\frac{4}{9} \)

9. \( s^8t \)

10. \( 8m^2n^4 \)

11. \( 9xy^3z^7 \)

12. \( -3q^4rs^6 \)

In Exercises 13–20, write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms. (See Examples 2 and 3.)

13. \( 6c^3 + 2e^4 - c \)

14. \( 4w^{11} - w^{12} \)

15. \( 7 + 3p^2 \)

16. \( 8d^2 - 2 - 4d^3 \)

17. \( 3r^8 \)

18. \( 5z^2 + 2z^3 + 3z^4 \)

19. \( \pi r^2 - \frac{4}{3} \pi r^3 + 2r^5 \)

20. \( \sqrt{7n^4} \)

21. **MODELING WITH MATHEMATICS** The expression \( \frac{4}{3} \pi r^3 \) represents the volume of a sphere with radius \( r \). Why is this expression a monomial? What is its degree?

22. **MODELING WITH MATHEMATICS** The amount of money you have after investing \$400 for 8 years and \$600 for 6 years at the same interest rate is represented by \( 400x^8 + 600x^6 \), where \( x \) is the growth factor. Classify the polynomial by the number of terms. What is its degree?

In Exercises 23–30, find the sum. (See Example 4.)

23. \( (5y + 4) + (-2y + 6) \)

24. \( (-8x - 12) + (9x + 4) \)

25. \( (2n^2 - 5n - 6) + (-n^2 - 3n + 11) \)

26. \( (-3p^3 + 5p^2 - 2p) + (-p^3 - 8p^2 - 15p) \)

27. \( (3g^2 - g) + (3g^2 - 8g + 4) \)

28. \( (9r^2 + 4r - 7) + (3r^2 - 3r) \)

29. \( (4a - a^3 - 3) + (2a^3 - 5a^2 + 8) \)

30. \( (s^3 - 2s - 9) + (2s^2 - 6s^3 + s) \)

In Exercises 31–38, find the difference. (See Example 5.)

31. \( (d - 9) - (3d - 1) \)

32. \( (6x + 9) - (7x + 1) \)

33. \( (y^2 - 4y + 9) - (3y^2 - 6y - 9) \)

34. \( (4m^2 - m + 2) - (-3m^2 + 10m + 4) \)

35. \( (k^3 - 7k + 2) - (k^3 - 12) \)

36. \( (-r - 10) - (-4r^3 + r^2 + 7r) \)
37. \((t^4 - t^2 + t) - (12 - 9t^2 - 7t)\)

38. \((4d - 6d^3 + 3d^2) - (10d^3 + 7d - 2)\)

**ERROR ANALYSIS** In Exercises 39 and 40, describe and correct the error in finding the sum or difference.

39. 

\[
(x^2 + x) - (2x^2 - 3x) = x^2 + x - 2x^2 + 3x
= (x^2 - 2x^2) + (x - 3x)
= -x^2 - 2x
\]

40. 

\[
x^3 - 4x^2 + 3
+ -3x^2 + 8x - 2
-2x^3 + 4x^2 + 1
\]

41. **MODELING WITH MATHEMATICS** The cost (in dollars) of making \(b\) bracelets is represented by \(4 + 5b\). The cost (in dollars) of making \(b\) necklaces is represented by \(8b + 6\). Write a polynomial that represents how much more it costs to make \(b\) necklaces than \(b\) bracelets.

42. **MODELING WITH MATHEMATICS** The number of individual memberships at a fitness center in \(m\) months is represented by \(142 + 12m\). The number of family memberships at the fitness center in \(m\) months is represented by \(52 + 6m\). Write a polynomial that represents the total number of memberships at the fitness center.

In Exercises 43–46, find the sum or difference.

43. \((2s^2 - 5st - r^2) - (s^2 + 7st - r^2)\)

44. \((a^2 - 3ab + 2b^2) + (-4a^2 + 5ab - b^2)\)

45. \((c^2 - 6d^2) + (c^2 - 2cd + 2d^2)\)

46. \((-x^2 + 9xy) - (x^2 + 6xy - 8y^2)\)

**REASONING** In Exercises 47–50, complete the statement with always, sometimes, or never. Explain your reasoning.

47. The terms of a polynomial are ________ monomials.

48. The difference of two trinomials is ________ a trinomial.

49. A binomial is ________ a polynomial of degree 2.

50. The sum of two polynomials is ________ a polynomial.

**MODELING WITH MATHEMATICS** The polynomial \(-16t^2 + v_0t + s_0\) represents the height (in feet) of an object, where \(v_0\) is the initial vertical velocity (in feet per second), \(s_0\) is the initial height of the object (in feet), and \(t\) is the time (in seconds). In Exercises 51 and 52, write a polynomial that represents the height of the object. Then find the height of the object after 1 second.

51. You throw a water balloon from a building.

52. You bounce a tennis ball on a racket.

53. **MODELING WITH MATHEMATICS** You drop a ball from a height of 98 feet. At the same time, your friend throws a ball upward. The polynomials represent the heights (in feet) of the balls after \(t\) seconds. (See Example 6.)

\[-16t^2 + 98\]

\[-16t^2 + 46t + 6\]

a. Before the balls reach the same height, write a polynomial that represents the distance between your ball and your friend’s ball after \(t\) seconds.

b. Interpret the coefficients of the polynomial in part (a).
54. **MODELING WITH MATHEMATICS** During a 7-year period, the amounts (in millions of dollars) spent each year on buying new vehicles \( N \) and used vehicles \( U \) by United States residents are modeled by the equations

\[
N = -0.028t^3 + 0.06t^2 + 0.1t + 17
\]

\[
U = -0.38t^2 + 1.5t + 42
\]

where \( t = 1 \) represents the first year in the 7-year period.

a. Write a polynomial that represents the total amount spent each year on buying new and used vehicles in the 7-year period.

b. How much is spent on buying new and used vehicles in the fifth year?

55. **MATHEMATICAL CONNECTIONS**

Write the polynomial in standard form that represents the perimeter of the quadrilateral.

![Quadrilateral Diagram]

56. **HOW DO YOU SEE IT?** The right side of the equation of each line is a polynomial.

\[
y = -2x + 1
\]

\[
y = x - 2
\]

a. The absolute value of the difference of the two polynomials represents the vertical distance between points on the lines with the same \( x \)-value. Write this expression.

b. When does the expression in part (a) equal 0? How does this value relate to the graph?

57. **MAKING AN ARGUMENT** Your friend says that when adding polynomials, the order in which you add does not matter. Is your friend correct? Explain.

58. **THOUGHT PROVOKING** Write two polynomials whose sum is \( x^2 \) and whose difference is 1.

59. **REASONING** Determine whether the set is closed under the given operation. Explain.

a. the set of negative integers; multiplication

b. the set of whole numbers; addition

60. **PROBLEM SOLVING** You are building a multi-level deck.

a. For each level, write a polynomial in standard form that represents the area of that level. Then write the polynomial in standard form that represents the total area of the deck.

b. What is the total area of the deck when \( x = 20 \)?

c. A gallon of deck sealant covers 400 square feet. How many gallons of sealant do you need to cover the deck in part (b) once? Explain.

61. **PROBLEM SOLVING** A hotel installs a new swimming pool and a new hot tub.

a. Write the polynomial in standard form that represents the area of the patio.

b. The patio will cost $10 per square foot. Determine the cost of the patio when \( x = 9 \).

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

62. \( 2(x - 1) + 3(x + 2) \)

63. \( 8(4y - 3) + 2(y - 5) \)

64. \( 5(2r + 1) - 3(-4r + 2) \)
7.2 Multiplying Polynomials

Essential Question
How can you multiply two polynomials?

**EXPLORATION 1** Multiplying Monomials Using Algebra Tiles

Work with a partner. Write each product. Explain your reasoning.

a. \( + \cdot + = \)

b. \( + \cdot - = \)

c. \( - \cdot - = \)

d. \( + \cdot + = \)

e. \( + \cdot - = \)

f. \( - \cdot + = \)

g. \( - \cdot - = \)

h. \( + \cdot + = \)

i. \( + \cdot - = \)

j. \( - \cdot - = \)

**EXPLORATION 2** Multiplying Binomials Using Algebra Tiles

Work with a partner. Write the product of two binomials modeled by each rectangular array of algebra tiles. In parts (c) and (d), first draw the rectangular array of algebra tiles that models each product.

a. \((x + 3)(x - 2) = \)

b. \((2x - 1)(2x + 1) = \)

c. \((x + 2)(2x - 1) = \)

d. \((-x - 2)(x - 3) = \)

Communicate Your Answer

3. How can you multiply two polynomials?

4. Give another example of multiplying two binomials using algebra tiles that is similar to those in Exploration 2.
What You Will Learn

- Multiply binomials.
- Use the FOIL Method.
- Multiply binomials and trinomials.

Multiplying Binomials

The product of two polynomials is always a polynomial. So, like the set of integers, the set of polynomials is closed under multiplication. You can use the Distributive Property to multiply two binomials.

**EXAMPLE 1** Multiplying Binomials Using the Distributive Property

Find (a) \((x + 2)(x + 5)\) and (b) \((x + 3)(x - 4)\).

**SOLUTION**

a. Use the horizontal method.

\[
(x + 2)(x + 5) = x(x + 5) + 2(x + 5)
\]
\[
= x(x) + x(5) + 2(x) + 2(5)
\]
\[
= x^2 + 5x + 2x + 10
\]
\[
= x^2 + 7x + 10
\]

- The product is \(x^2 + 7x + 10\).

b. Use the vertical method.

\[
\begin{array}{c}
\phantom{x + 3} \\
\times \quad x - 4
\end{array}
\]

\[
\begin{array}{c}
4x + 12 \\
3x - 12
\end{array}
\]

\[
\begin{array}{c}
x^2 + 3x \\
x^2 - x - 12
\end{array}
\]

- The product is \(x^2 - x - 12\).

**EXAMPLE 2** Multiplying Binomials Using a Table

Find \((2x - 3)(x + 5)\).

**SOLUTION**

Step 1. Write each binomial as a sum of terms.

\[
(2x - 3)(x + 5) = [2x + (-3)](x + 5)
\]

Step 2. Make a table of products.

\[
\begin{array}{c|c|c}
2x & -3 \\
\hline
x & 2x^2 & -3x \\
5 & 10x & -15
\end{array}
\]

- The product is \(2x^2 - 3x + 10x - 15\), or \(2x^2 + 7x - 15\).

**Monitoring Progress**

Use the Distributive Property to find the product.

1. \((y + 4)(y + 1)\)

2. \((z - 2)(z + 6)\)

Use a table to find the product.

3. \((p + 3)(p - 8)\)

4. \((r - 5)(2r - 1)\)
Using the FOIL Method

The **FOIL Method** is a shortcut for multiplying two binomials.

### Core Concept

**FOIL Method**

To multiply two binomials using the FOIL Method, find the sum of the products of the

- **First terms**, \((x + 1)(x + 2)\)
- **Outer terms**, \((x + 1)(x + 2)\)
- **Inner terms**, \((x + 1)(x + 2)\)
- **Last terms**, \((x + 1)(x + 2)\)

\((x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2\)

### Example 3

**Multiplying Binomials Using the FOIL Method**

Find each product.

**a.** \((x - 3)(x - 6)\)

**b.** \((2x + 1)(3x - 5)\)

**SOLUTION**

**a.** Use the FOIL Method.

\[
(x - 3)(x - 6) = x(x) + x(-6) + (-3)(x) + (-3)(-6)
\]

\[
= x^2 - 6x - 3x + 18
\]

\[
= x^2 - 9x + 18
\]

The product is \(x^2 - 9x + 18\).

**b.** Use the FOIL Method.

\[
(2x + 1)(3x - 5) = 2x(3x) + 2x(-5) + 1(3x) + 1(-5)
\]

\[
= 6x^2 - 10x + 3x - 5
\]

\[
= 6x^2 - 7x - 5
\]

The product is \(6x^2 - 7x - 5\).

### Monitoring Progress

Use the FOIL Method to find the product.

5. \((m - 3)(m - 7)\)

6. \((x - 4)(x + 2)\)

7. \((2u + \frac{1}{2})(u - \frac{3}{2})\)

8. \((n + 2)(n^2 + 3)\)
Multiplying Binomials and Trinomials

**Example 4**  Multiplying a Binomial and a Trinomial

Find \((x + 5)(x^2 - 3x - 2)\).

**Solution**

\[
\begin{align*}
\begin{array}{c}
\text{Align like terms vertically.} \\
\text{Distribute Property} \\
\text{Combine like terms.}
\end{array}
\end{align*}
\]

The product is \(x^3 + 2x^2 - 17x - 10\).

**Example 5**  Solving a Real-Life Problem

In hockey, a goalie behind the goal line can only play a puck in the trapezoidal region.

a. Write a polynomial that represents the area of the trapezoidal region.

b. Find the area of the trapezoidal region when the shorter base is 18 feet.

**Solution**

a. \(\frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(x - 7)[x + (x + 10)]\)

\[= \frac{1}{2}(x - 7)(2x + 10)\]

\[= \frac{1}{2}[2x^2 + 10x + (-14x) + (-70)]\]

\[= \frac{1}{2}(2x^2 - 4x - 70)\]

\[= x^2 - 2x - 35\]

A polynomial that represents the area of the trapezoidal region is \(x^2 - 2x - 35\).

b. Find the value of \(x^2 - 2x - 35\) when \(x = 18\).

\[x^2 - 2x - 35 = 18^2 - 2(18) - 35\]

\[= 324 - 36 - 35\]

\[= 253\]

The area of the trapezoidal region is 253 square feet.

---

**Monitoring Progress**

Find the product.

9. \((x + 1)(x^2 + 5x + 8)\)

10. \((n - 3)(n^2 - 2n + 4)\)

11. **WHAT IF?** In Example 5(a), how does the polynomial change when the longer base is extended by 1 foot? Explain.
7.2 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** Describe two ways to find the product of two binomials.

2. **WRITING** Explain how the letters of the word FOIL can help you to remember how to multiply two binomials.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, use the Distributive Property to find the product. *(See Example 1.)*

3. \((x + 1)(x + 3)\)
4. \((y + 6)(y - 4)\)
5. \((z - 5)(z + 3)\)
6. \((a + 8)(a - 3)\)
7. \((g - 7)(g - 2)\)
8. \((n - 6)(n + 4)\)
9. \((3m + 1)(m + 9)\)
10. \((5x + 6)(x - 2)\)

In Exercises 11–18, use a table to find the product. *(See Example 2.)*

11. \((x + 3)(x + 2)\)
12. \((y + 10)(y - 5)\)
13. \((h - 8)(h - 9)\)
14. \((c - 6)(c - 5)\)
15. \((3k - 1)(4k + 9)\)
16. \((5g + 3)(g + 8)\)
17. \((-3 + 2j)(4j - 7)\)
18. \((5d - 12)(-7 + 3d)\)

**ERROR ANALYSIS** In Exercises 19 and 20, describe and correct the error in finding the product of the binomials.

19. \((t - 2)(t + 5) = t - 2(t + 5)\)
   \[= t - 2t - 10\]
   \[= -t - 10\]

20. \((x - 5)(3x + 1)\)
   \[
   \begin{array}{c|c|c|c|c|c|c|c}
   x & 3x^2 & x & 5x & 15x & 5 \\
   \hline
   \end{array}
   \]
   \[(x - 5)(3x + 1) = 3x^2 + 16x + 5\]

In Exercises 21–30, use the FOIL Method to find the product. *(See Example 3.)*

21. \((b + 3)(b + 7)\)
22. \((w + 9)(w + 6)\)
23. \((k + 5)(k - 1)\)
24. \((x - 4)(x + 8)\)
25. \((g - \frac{1}{3})(g + \frac{1}{3})\)
26. \((z - \frac{5}{3})(z - \frac{5}{3})\)
27. \((9 - r)(2 - 3r)\)
28. \((8 - 4x)(2x + 6)\)
29. \((w + 5)(w^2 + 3w)\)
30. \((v - 3)(v^2 + 8v)\)

**MATHEMATICAL CONNECTIONS** In Exercises 31–34, write a polynomial that represents the area of the shaded region.

31. \(2x - 9\)
32. \(x + 5\)
33. \(x + 6\)
34. \(x + 1\)

In Exercises 35–42, find the product. *(See Example 4.)*

35. \((x + 4)(x^2 + 3x + 2)\)
36. \((f + 1)(f^2 + 4f + 8)\)
37. \((y + 3)(y^2 + 8y - 2)\)
38. \((t - 2)(t^2 - 5t + 1)\)
39. \((4 - b)(5b^2 + 5b - 4)\)
40. \((d + 6)(2d^2 - d + 7)\)
41. \((3e^2 - 5e + 7)(6e + 1)\)
42. \((6v^2 + 2v - 9)(4 - 5v)\)
43. **MODELING WITH MATHEMATICS** The football field is rectangular. (See Example 5.)

\[(10x + 10) \text{ ft} \quad (4x + 20) \text{ ft}\]

a. Write a polynomial that represents the area of the football field.

b. Find the area of the football field when the width is 160 feet.

44. **MODELING WITH MATHEMATICS** You design a frame to surround a rectangular photo. The width of the frame is the same on every side, as shown.

![Frame Diagram]

a. Write a polynomial that represents the combined area of the photo and the frame.

b. Find the combined area of the photo and the frame when the width of the frame is 4 inches.

45. **WRITING** When multiplying two binomials, explain how the degree of the product is related to the degree of each binomial.

46. **THOUGHT PROVOKING** Write two polynomials that are not monomials whose product is a trinomial of degree 3.

47. **MAKING AN ARGUMENT** Your friend says the FOIL Method can be used to multiply two trinomials. Is your friend correct? Explain your reasoning.

48. **HOW DO YOU SEE IT?** The table shows one method of finding the product of two binomials.

<table>
<thead>
<tr>
<th></th>
<th>$-4x$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-8x$</td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$-9$</td>
<td>$c$</td>
<td>$d$</td>
</tr>
</tbody>
</table>

a. Write the two binomials being multiplied.

b. Determine whether $a$, $b$, $c$, and $d$ will be positive or negative when $x > 0$.

49. **COMPARING METHODS** You use the Distributive Property to multiply $(x + 3)(x - 5)$. Your friend uses the FOIL Method to multiply $(x - 5)(x + 3)$. Should your answers be equivalent? Justify your answer.

50. **USING STRUCTURE** The shipping container is a rectangular prism. Write a polynomial that represents the volume of the container.

\[(4x - 3) \text{ ft} \quad (x + 1) \text{ ft} \quad (x + 2) \text{ ft}\]

51. **ABSTRACT REASONING** The product of $(x + m)(x + n)$ is $x^2 + bx + c$.

a. What do you know about $m$ and $n$ when $c > 0$?

b. What do you know about $m$ and $n$ when $c < 0$?

52. Write the absolute value function as a piecewise function. (Section 4.7)

\[y = |x| + 4\]

53. \[y = 6|x - 3|\]

54. \[y = -4|x + 2|\]

55. Simplify the expression. Write your answer using only positive exponents. (Section 6.1)

\[10^2 \cdot 10^9\]

56. \[\frac{x^5 \cdot x}{x^8}\]

57. \[(3x^2)^{-3}\]

58. \[\left(\frac{2y^4}{y^3}\right)^{-2}\]
7.3 Special Products of Polynomials

Essential Question What are the patterns in the special products \((a + b)(a - b)\), \((a + b)^2\), and \((a - b)^2\)?

**EXPLORATION 1** Finding a Sum and Difference Pattern

Work with a partner. Write the product of two binomials modeled by each rectangular array of algebra tiles.

a. \((x + 2)(x - 2) = \)

b. \((2x - 1)(2x + 1) = \)

![Algebra tiles for a and b]

**EXPLORATION 2** Finding the Square of a Binomial Pattern

Work with a partner. Draw the rectangular array of algebra tiles that models each product of two binomials. Write the product.

a. \((x + 2)^2 = \)

b. \((2x - 1)^2 = \)

![Algebra tiles for a and b]

Communicate Your Answer

3. What are the patterns in the special products \((a + b)(a - b)\), \((a + b)^2\), and \((a - b)^2\)?

4. Use the appropriate special product pattern to find each product. Check your answers using algebra tiles.

a. \((x + 3)(x - 3)\)

b. \((x - 4)(x + 4)\)

c. \((3x + 1)(3x - 1)\)

d. \((x + 3)^2\)

e. \((x - 2)^2\)

f. \((3x + 1)^2\)
What You Will Learn

- Use the square of a binomial pattern.
- Use the sum and difference pattern.
- Use special product patterns to solve real-life problems.

Using the Square of a Binomial Pattern

The diagram shows a square with a side length of \((a + b)\) units. You can see that the area of the square is 
\[(a + b)^2 = a^2 + 2ab + b^2.\]

This is one version of a pattern called the square of a binomial. To find another version of this pattern, use algebra: replace \(b\) with \(-b\).

\[(a + (-b))^2 = a^2 + 2a(-b) + (-b)^2\]
\[(a - b)^2 = a^2 - 2ab + b^2\]

Replace \(b\) with \(-b\) in the pattern above.

\[(a - b)^2 = a^2 - 2ab + b^2\]
Simplify.

Core Concept

Square of a Binomial Pattern

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a + b)^2 = a^2 + 2ab + b^2)</td>
<td>((x + 5)^2 = (x)^2 + 2(x)(5) + (5)^2)</td>
</tr>
<tr>
<td></td>
<td>[= x^2 + 10x + 25]</td>
</tr>
<tr>
<td>((a - b)^2 = a^2 - 2ab + b^2)</td>
<td>((2x - 3)^2 = (2x)^2 - 2(2x)(3) + (3)^2)</td>
</tr>
<tr>
<td></td>
<td>[= 4x^2 - 12x + 9]</td>
</tr>
</tbody>
</table>

EXAMPLE 1 Using the Square of a Binomial Pattern

Find each product.

a. \((3x + 4)^2\)

SOLUTION

a. \((3x + 4)^2 = (3x)^2 + 2(3x)(4) + 4^2\] \[= 9x^2 + 24x + 16\]
Square of a binomial pattern
Simplify.

The product is \(9x^2 + 24x + 16\).

b. \((5x - 2y)^2\)

SOLUTION

b. \((5x - 2y)^2 = (5x)^2 - 2(5x)(2y) + (2y)^2\] \[= 25x^2 - 20xy + 4y^2\]
Square of a binomial pattern
Simplify.

The product is \(25x^2 - 20xy + 4y^2\).

Monitoring Progress

Help in English and Spanish at BigIdeasMath.com

Find the product.

1. \((x + 7)^2\)
2. \((7x - 3)^2\)
3. \((4x - y)^2\)
4. \((3m + n)^2\)
Using the Sum and Difference Pattern

To find the product \((x + 2)(x - 2)\), you can multiply the two binomials using the FOIL Method.

\[
(x + 2)(x - 2) = x^2 - 2x + 2x - 4 \quad \text{FOIL Method}
\]
\[
= x^2 - 4 \quad \text{Combine like terms.}
\]

This suggests a pattern for the product of the sum and difference of two terms.

**Core Concept**

**Sum and Difference Pattern**

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a + b)(a - b) = a^2 - b^2)</td>
<td>((x + 3)(x - 3) = x^2 - 9)</td>
</tr>
</tbody>
</table>

**EXAMPLE 2** Using the Sum and Difference Pattern

Find each product.

a. \((t + 5)(t - 5)\)  
b. \((3x + y)(3x - y)\)

**SOLUTION**

a. \((t + 5)(t - 5) = t^2 - 5^2\)  
   \[= t^2 - 25\]
   The product is \(t^2 - 25\).  

b. \((3x + y)(3x - y) = (3x)^2 - y^2\)  
   \[= 9x^2 - y^2\]
   The product is \(9x^2 - y^2\).

The special product patterns can help you use mental math to find certain products of numbers.

**EXAMPLE 3** Using Special Product Patterns and Mental Math

Use special product patterns to find the product \(26 \cdot 34\).

**SOLUTION**

Notice that 26 is 4 less than 30, while 34 is 4 more than 30.

\[26 \cdot 34 = (30 - 4)(30 + 4)\]  
Write as product of difference and sum.
\[= 30^2 - 4^2\]  
Sum and difference pattern
\[= 900 - 16\]  
Evaluate powers.
\[= 884\]  
Simplify.

The product is 884.

**Monitoring Progress**

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Find the product.

5. \((x + 10)(x - 10)\)  
6. \((2x + 1)(2x - 1)\)  
7. \((x + 3y)(x - 3y)\)

8. Describe how to use special product patterns to find \(21^2\).
Solving Real-Life Problems

**EXAMPLE 4** Modeling with Mathematics

A combination of two genes determines the color of the dark patches of a border collie’s coat. An offspring inherits one patch color gene from each parent. Each parent has two color genes, and the offspring has an equal chance of inheriting either one.

The gene $B$ is for black patches, and the gene $r$ is for red patches. Any gene combination with a $B$ results in black patches. Suppose each parent has the same gene combination $Br$. The Punnett square shows the possible gene combinations of the offspring and the resulting patch colors.

a. What percent of the possible gene combinations result in black patches?

b. Show how you could use a polynomial to model the possible gene combinations.

**SOLUTION**

a. Notice that the Punnett square shows four possible gene combinations of the offspring. Of these combinations, three result in black patches.

So, $75\%$ of the possible gene combinations result in black patches.

b. Model the gene from each parent with $0.5B + 0.5r$. There is an equal chance that the offspring inherits a black or a red gene from each parent.

You can model the possible gene combinations of the offspring with $(0.5B + 0.5r)^2$. Notice that this product also represents the area of the Punnett square.

Expand the product to find the possible patch colors of the offspring.

$$(0.5B + 0.5r)^2 = (0.5B)^2 + 2(0.5B)(0.5r) + (0.5r)^2$$

$$= 0.25B^2 + 0.5Br + 0.25r^2$$

Consider the coefficients in the polynomial.

The coefficients show that $25\% + 50\% = 75\%$ of the possible gene combinations result in black patches.

**Monitoring Progress**

9. Each of two dogs has one black gene ($B$) and one white gene ($W$). The Punnett square shows the possible gene combinations of an offspring and the resulting colors.

a. What percent of the possible gene combinations result in black?

b. Show how you could use a polynomial to model the possible gene combinations of the offspring.
7.3 Exercises

Vocabulary and Core Concept Check

1. WRITING Explain how to use the square of a binomial pattern.

2. WHICH ONE DOESN’T BELONG? Which expression does not belong with the other three? Explain your reasoning.
   - (x + 1)(x - 1)
   - (3x + 2)(3x - 2)
   - (x + 2)(x - 3)
   - (2x + 5)(2x - 5)

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the product. (See Example 1.)

3. \((x + 8)^2\)
4. \((a - 6)^2\)
5. \((2f - 1)^2\)
6. \((5p + 2)^2\)
7. \((-7t + 4)^2\)
8. \((-12 - n)^2\)
9. \((2a + b)^2\)
10. \((6x - 3y)^2\)

MATHEMATICAL CONNECTIONS In Exercises 11–14, write a polynomial that represents the area of the square.

11. \(x^2 + 4x + 4\)
12. \(x^2 + 7x + 49\)
13. \((7n - 5)^2\)
14. \((4c + 4d)^2\)

In Exercises 15–24, find the product. (See Example 2.)

15. \((t - 7)(t + 7)\)
16. \((m + 6)(m - 6)\)
17. \((4x + 1)(4x - 1)\)
18. \((2k - 4)(2k + 4)\)
19. \((8 + 3a)(8 - 3a)\)
20. \((\frac{1}{2} - c)(\frac{1}{2} + c)\)
21. \((p - 10q)(p + 10q)\)
22. \((7m + 8n)(7m - 8n)\)
23. \((-y + 4)(-y - 4)\)
24. \((-5g - 2h)(-5g + 2h)\)

In Exercises 25–30, use special product patterns to find the product. (See Example 3.)

25. \(16 \cdot 24\)
26. \(33 \cdot 27\)
27. \(42^2\)
28. \(29^2\)
29. \(30.5^2\)
30. \(10\frac{1}{2} \cdot 9\frac{2}{3}\)

ERROR ANALYSIS In Exercises 31 and 32, describe and correct the error in finding the product.

31. \((k + 4)^2 = k^2 + 4^2 = k^2 + 16\)
32. \((s + 5)(s - 5) = s^2 + 2(s)(5) - 5^2 = s^2 + 10s - 25\)

33. MODELING WITH MATHEMATICS A contractor extends a house on two sides.

a. The area of the house after the renovation is represented by \((x + 50)^2\). Find this product.
b. Use the polynomial in part (a) to find the area when \(x = 15\). What is the area of the extension?
34. **MODELING WITH MATHEMATICS** A square-shaped parking lot with 100-foot sides is reduced by \(x\) feet on one side and extended by \(x\) feet on an adjacent side.

a. The area of the new parking lot is represented by \((100 - x)(100 + x)\). Find this product.

b. Does the area of the parking lot increase, decrease, or stay the same? Explain.

c. Use the polynomial in part (a) to find the area of the new parking lot when \(x = 21\).

35. **MODELING WITH MATHEMATICS** In deer, the gene \(N\) is for normal coloring and the gene \(a\) is for no coloring, or albino. Any gene combination with an \(N\) results in normal coloring. The Punnett square shows the possible gene combinations of an offspring and the resulting colors from parents that both have the gene combination \(Na\). (See Example 4.)

a. What percent of the possible gene combinations result in albino coloring?

b. Show how you could use a polynomial to model the possible gene combinations of the offspring.

36. **MODELING WITH MATHEMATICS** Your iris controls the amount of light that enters your eye by changing the size of your pupil.

a. Write a polynomial that represents the area of your pupil. Write your answer in terms of \(\pi\).

b. The width \(x\) of your iris decreases from 4 millimeters to 2 millimeters when you enter a dark room. How many times greater is the area of your pupil after entering the room than before entering the room? Explain.

37. **CRITICAL THINKING** Write two binomials that have the product \(x^2 - 121\). Explain.

38. **HOW DO YOU SEE IT?** In pea plants, any gene combination with a green gene (\(G\)) results in a green pod. The Punnett square shows the possible gene combinations of the offspring of two \(Gy\) pea plants and the resulting pod colors.

A polynomial that models the possible gene combinations of the offspring is

\[(0.5G + 0.5y)^2 = 0.25G^2 + 0.5Gy + 0.25y^2.\]

Describe two ways to determine the percent of possible gene combinations that result in green pods.

In Exercises 39–42, find the product.

39. \((x^2 + 1)(x^2 - 1)\)  
40. \((y^3 + 4)^2\)

41. \((2m^2 - 3n^3)^2\)  
42. \((r^3 - 6r^4)(r^3 + 6r^4)\)

43. **MAKING AN ARGUMENT** Your friend claims to be able to use a special product pattern to determine that \((4 \frac{1}{2})^2\) is equal to \(16 \frac{1}{4}\). Is your friend correct? Explain.

44. **THOUGHT PROVOKING** The area (in square meters) of the surface of an artificial lake is represented by \(x^2\). Describe three ways to modify the dimensions of the lake so that the new area can be represented by the three types of special product patterns discussed in this section.

45. **REASONING** Find \(k\) so that \(9x^2 - 48x + k\) is the square of a binomial.

46. **REPEATED REASONING** Find \((x + 1)^3\) and \((x + 2)^3\). Find a pattern in the terms and use it to write a pattern for the cube of a binomial \((a + b)^3\).

47. **PROBLEM SOLVING** Find two numbers \(a\) and \(b\) such that \((a + b)(a - b) < (a - b)^2 < (a + b)^2\).

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Factor the expression using the GCF. (Skills Review Handbook)

48. \(12y - 18\)  
49. \(9r + 27\)  
50. \(49x + 35t\)  
51. \(15x - 10y\)
Essential Question

How can you solve a polynomial equation?

EXPLORATION 1  Matching Equivalent Forms of an Equation

Work with a partner. An equation is considered to be in **factored form** when the product of the factors is equal to 0. Match each factored form of the equation with its equivalent standard form and nonstandard form.

<table>
<thead>
<tr>
<th>Factored Form</th>
<th>Standard Form</th>
<th>Nonstandard Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ((x - 1)(x - 3) = 0)</td>
<td>A. (x^2 - x - 2 = 0)</td>
<td>1. (x^2 - 5x = -6)</td>
</tr>
<tr>
<td>b. ((x - 2)(x - 3) = 0)</td>
<td>B. (x^2 + x - 2 = 0)</td>
<td>2. ((x - 1)^2 = 4)</td>
</tr>
<tr>
<td>c. ((x + 1)(x - 2) = 0)</td>
<td>C. (x^2 - 4x + 3 = 0)</td>
<td>3. (x^2 - x = 2)</td>
</tr>
<tr>
<td>d. ((x - 1)(x + 2) = 0)</td>
<td>D. (x^2 - 5x + 6 = 0)</td>
<td>4. (x(x + 1) = 2)</td>
</tr>
<tr>
<td>e. ((x + 1)(x - 3) = 0)</td>
<td>E. (x^2 - 2x - 3 = 0)</td>
<td>5. (x^2 - 4x = -3)</td>
</tr>
</tbody>
</table>

EXPLORATION 2  Writing a Conjecture

Work with a partner. Substitute 1, 2, 3, 4, 5, and 6 for \(x\) in each equation and determine whether the equation is true. Organize your results in a table. Write a conjecture describing what you discovered.

| a. \((x - 1)(x - 2) = 0\) | b. \((x - 2)(x - 3) = 0\) |
| c. \((x - 3)(x - 4) = 0\) | d. \((x - 4)(x - 5) = 0\) |
| e. \((x - 5)(x - 6) = 0\) | f. \((x - 6)(x - 1) = 0\) |

EXPLORATION 3  Special Properties of 0 and 1

Work with a partner. The numbers 0 and 1 have special properties that are shared by no other numbers. For each of the following, decide whether the property is true for 0, 1, both, or neither. Explain your reasoning.

a. When you add [ ] to a number \(n\), you get \(n\).

b. If the product of two numbers is [ ], then at least one of the numbers is 0.

c. The square of [ ] is equal to itself.

d. When you multiply a number \(n\) by [ ], you get \(n\).

e. When you multiply a number \(n\) by [ ], you get 0.

f. The opposite of [ ] is equal to itself.

Communicate Your Answer

4. How can you solve a polynomial equation?

5. One of the properties in Exploration 3 is called the **Zero-Product Property**. It is one of the most important properties in all of algebra. Which property is it? Why do you think it is called the Zero-Product Property? Explain how it is used in algebra and why it is so important.
What You Will Learn

- Use the Zero-Product Property.
- Factor polynomials using the GCF.
- Use the Zero-Product Property to solve real-life problems.

Using the Zero-Product Property
A polynomial is in **factored form** when it is written as a product of factors.

<table>
<thead>
<tr>
<th>Standard form</th>
<th>Factored form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 2x )</td>
<td>( x(x + 2) )</td>
</tr>
<tr>
<td>( x^2 + 5x - 24 )</td>
<td>( (x - 3)(x + 8) )</td>
</tr>
</tbody>
</table>

When one side of an equation is a polynomial in factored form and the other side is 0, use the **Zero-Product Property** to solve the polynomial equation. The solutions of a polynomial equation are also called **roots**.

**Core Concept**

**Zero-Product Property**

**Words**
If the product of two real numbers is 0, then at least one of the numbers is 0.

**Algebra**
If \( a \) and \( b \) are real numbers and \( ab = 0 \), then \( a = 0 \) or \( b = 0 \).

**EXAMPLE 1** Solving Polynomial Equations

Solve each equation.

**a.** \( 2x(x - 4) = 0 \)

**b.** \( (x - 3)(x - 9) = 0 \)

**SOLUTION**

**a.** \( 2x(x - 4) = 0 \)

Write equation.

\[ 2x = 0 \quad \text{or} \quad x - 4 = 0 \]

Zero-Product Property

\[ x = 0 \quad \text{or} \quad x = 4 \]

Solve for \( x \).

The roots are \( x = 0 \) and \( x = 4 \).

**b.** \( (x - 3)(x - 9) = 0 \)

Write equation.

\[ x - 3 = 0 \quad \text{or} \quad x - 9 = 0 \]

Zero-Product Property

\[ x = 3 \quad \text{or} \quad x = 9 \]

Solve for \( x \).

The roots are \( x = 3 \) and \( x = 9 \).

**Monitoring Progress**

Solve the equation. Check your solutions.

1. \( x(x - 1) = 0 \)
2. \( 3(t + 2) = 0 \)
3. \( (z - 4)(z - 6) = 0 \)
When two or more roots of an equation are the same number, the equation has **repeated roots**.

**Solving Polynomial Equations**

Solve each equation.

a. \((2x + 7)(2x - 7) = 0\)

**SOLUTION**

\[
2x + 7 = 0 \quad \text{or} \quad 2x - 7 = 0
\]

\[
x = -\frac{7}{2} \quad \text{or} \quad x = \frac{7}{2}
\]

The roots are \(x = -\frac{7}{2}\) and \(x = \frac{7}{2}\).

b. \((x - 1)\)\((x - 1) = 0\)

**SOLUTION**

\[
x - 1 = 0 \quad \text{or} \quad x - 1 = 0
\]

\[
x = 1 \quad \text{or} \quad x = 1
\]

The equation has repeated roots of \(x = 1\).

c. \((x + 1)(x - 3)(x - 2) = 0\)

**SOLUTION**

\[
x + 1 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x - 2 = 0
\]

\[
x = -1 \quad \text{or} \quad x = 3 \quad \text{or} \quad x = 2
\]

The roots are \(x = -1\), \(x = 3\), and \(x = 2\).

**Factoring Polynomials Using the GCF**

To solve a polynomial equation using the Zero-Product Property, you may need to **factor** the polynomial, or write it as a product of other polynomials. Look for the **greatest common factor** (GCF) of the terms of the polynomial. This is a monomial that divides evenly into each term.

**Finding the Greatest Common Monomial Factor**

Factor out the greatest common monomial factor from \(4x^4 + 24x^3\).

**SOLUTION**

The GCF of \(4\) and \(24\) is \(4\). The GCF of \(x^4\) and \(x^3\) is \(x^3\). So, the greatest common monomial factor of the terms is \(4x^3\).

\[
4x^4 + 24x^3 = 4x^3(x + 6).
\]
Solve (a) \(2x^2 + 8x = 0\) and (b) \(6n^2 = 15n\).

**SOLUTION**

**a.** \(2x^2 + 8x = 0\)

Write equation.

\[2x(x + 4) = 0\]

Factor left side.

\[2x = 0 \quad \text{or} \quad x + 4 = 0\]

Zero-Product Property

\[x = 0 \quad \text{or} \quad x = -4\]

Solve for \(x\).

The roots are \(x = 0\) and \(x = -4\).

**b.** \(6n^2 = 15n\)

Write equation.

\[6n^2 - 15n = 0\]

Subtract 15n from each side.

\[3n(2n - 5) = 0\]

Factor left side.

\[3n = 0 \quad \text{or} \quad 2n - 5 = 0\]

Zero-Product Property

\[n = 0 \quad \text{or} \quad n = \frac{5}{2}\]

Solve for \(n\).

The roots are \(n = 0\) and \(n = \frac{5}{2}\).

---

**Solving Real-Life Problems**

**EXAMPLE 5** Modeling with Mathematics

You can model the arch of a fireplace using the equation \(y = -\frac{1}{2}(x + 18)(x - 18)\), where \(x\) and \(y\) are measured in inches. The \(x\)-axis represents the floor. Find the width of the arch at floor level.

**SOLUTION**

Use the \(x\)-coordinates of the points where the arch meets the floor to find the width. At floor level, \(y = 0\). So, substitute 0 for \(y\) and solve for \(x\).

\[y = -\frac{1}{2}(x + 18)(x - 18)\]

Write equation.

\[0 = -\frac{1}{2}(x + 18)(x - 18)\]

Substitute 0 for \(y\).

\[0 = (x + 18)(x - 18)\]

Multiply each side by \(-9\).

\[x + 18 = 0 \quad \text{or} \quad x - 18 = 0\]

Zero-Product Property

\[x = -18 \quad \text{or} \quad x = 18\]

Solve for \(x\).

The width is the distance between the \(x\)-coordinates, \(-18\) and 18.

So, the width of the arch at floor level is \(\left|-18 - 18\right| = 36\) inches.

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11. You can model the entrance to a mine shaft using the equation \(y = -\frac{1}{2}(x + 4)(x - 4)\), where \(x\) and \(y\) are measured in feet. The \(x\)-axis represents the ground. Find the width of the entrance at ground level.
7.4 Exercises

Vocabulary and Core Concept Check

1. **WRITING** Explain how to use the Zero-Product Property to find the solutions of the equation $3x(x - 6) = 0$.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find both answers.

   - Solve the equation $(2k + 4)(k - 3) = 0$.
   - Find the value of $k$ for which $(2k + 4) + (k - 3) = 0$.
   - Find the value of $k$ for which $(2k + 4)(k - 3) = 0$.
   - Find the roots of the equation $(2k + 4)(k - 3) = 0$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, solve the equation. (See Example 1.)

3. $x(x + 7) = 0$
4. $r(r - 10) = 0$
5. $12t(t - 5) = 0$
6. $-2y(y + 1) = 0$
7. $(s - 9)(s - 1) = 0$
8. $(y + 2)(y - 6) = 0$

In Exercises 9–20, solve the equation. (See Example 2.)

9. $(2a - 6)(3a + 15) = 0$
10. $(4q + 3)(q + 2) = 0$
11. $(5m + 4)^2 = 0$
12. $(h - 8)^2 = 0$
13. $(3 - 2g)(7 - g) = 0$
14. $(2 - 4d)(2 + 4d) = 0$
15. $(z + 2)(z - 1) = 0$
16. $5p(2p - 3)(p + 7) = 0$
17. $(r - 4)^2(r + 8) = 0$
18. $w(w - 6)^2 = 0$
19. $(15 - 5c)(5c + 5) = (c + 6) = 0$
20. $(2 - n)(6 + 2n)(n + 2) = 0$

In Exercises 21–24, find the x-coordinates of the points where the graph crosses the x-axis.

21. $y = (x - 8)(x + 8)$
22. $y = (x + 1)(x + 7)$

23. $y = -(x - 14)(x - 5)$
24. $y = -0.2(x + 22)(x - 15)$

In Exercises 25–30, factor the polynomial. (See Example 3.)

25. $5z^2 + 45z$
26. $6d^2 - 21d$
27. $3y^3 - 9y^2$
28. $20x^3 + 30x^2$
29. $5n^6 + 2n^5$
30. $12a^4 + 8a$

In Exercises 31–36, solve the equation. (See Example 4.)

31. $4p^2 - p = 0$
32. $6m^2 + 12m = 0$
33. $25c + 10c^2 = 0$
34. $18q - 2q^2 = 0$
35. $3n^2 = 9n$
36. $-28r = 4r^2$

37. **ERROR ANALYSIS** Describe and correct the error in solving the equation.
   
   $6x(x + 5) = 0$
   $x + 5 = 0$
   $x = -5$
   The root is $x = -5$.
38. **ERROR ANALYSIS** Describe and correct the error in solving the equation.

\[3y^2 = 21y\]
\[3y = 21\]
\[y = 7\]

The root is \(y = 7\).

✗

39. **MODELING WITH MATHEMATICS** The entrance of a tunnel can be modeled by \(y = -\frac{11}{50}(x - 4)(x - 24)\), where \(x\) and \(y\) are measured in feet. The \(x\)-axis represents the ground. Find the width of the tunnel at ground level. (See Example 5.)

40. **MODELING WITH MATHEMATICS** The Gateway Arch in St. Louis can be modeled by \(y = -\frac{2}{315}(x + 315)(x - 315)\), where \(x\) and \(y\) are measured in feet. The \(x\)-axis represents the ground.

41. **MODELING WITH MATHEMATICS** A penguin leaps out of the water while swimming. This action is called porpoising. The height \(y\) (in feet) of a porpoising penguin can be modeled by \(y = -16x^2 + 4.8x\), where \(x\) is the time (in seconds) since the penguin leaped out of the water. Find the roots of the equation when \(y = 0\). Explain what the roots mean in this situation.

42. **HOW DO YOU SEE IT?** Use the graph to fill in each blank in the equation with the symbol + or −. Explain your reasoning.

\[y = (x - \underline{5})(x - \underline{3})\]

43. **CRITICAL THINKING** How many \(x\)-intercepts does the graph of \(y = (2x + 5)(x - 9)^2\) have? Explain.

44. **MAKING AN ARGUMENT** Your friend says that the graph of the equation \(y = (x - a)(x - b)\) always has two \(x\)-intercepts for any values of \(a\) and \(b\). Is your friend correct? Explain.

45. **CRITICAL THINKING** Does the equation \((x^2 + 3)(x^2 + 1) = 0\) have any real roots? Explain.

46. **THOUGHT PROVOKING** Write a polynomial equation of degree 4 whose only roots are \(x = 1\), \(x = 2\), and \(x = 3\).

47. **REASONING** Find the values of \(x\) in terms of \(y\) that are solutions of each equation.

a. \((x + y)(2x - y) = 0\)

b. \((x^2 - y^2)(4x + 16y) = 0\)

48. **PROBLEM SOLVING** Solve the equation \((4x^5 - 16)(3x^7 - 81) = 0\).

Maintaining Mathematical Proficiency

List the factor pairs of the number. (Skills Review Handbook)

49. 10
50. 18
51. 30
52. 48
7.1–7.4 What Did You Learn?

Core Vocabulary

- monomial, p. 358
- degree of a monomial, p. 358
- polynomial, p. 359
- binomial, p. 359
- trinomial, p. 359
- degree of a polynomial, p. 359
- standard form, p. 359
- leading coefficient, p. 359
- closed, p. 360
- FOIL Method, p. 367
- factored form, p. 378
- Zero-Product Property, p. 378
- roots, p. 378
- repeated roots, p. 379

Core Concepts

Section 7.1
- Polynomials, p. 359
- Adding Polynomials, p. 360
- Subtracting Polynomials, p. 360

Section 7.2
- Multiplying Binomials, p. 366
- FOIL Method, p. 367
- Multiplying Binomials and Trinomials, p. 368

Section 7.3
- Square of a Binomial Pattern, p. 372
- Sum and Difference Pattern, p. 373

Section 7.4
- Zero-Product Property, p. 378
- Factoring Polynomials Using the GCF, p. 379

Mathematical Practices

1. Explain how you wrote the polynomial in Exercise 11 on page 375. Is there another method you can use to write the same polynomial?

2. Find a shortcut for exercises like Exercise 7 on page 381 when the variable has a coefficient of 1. Does your shortcut work when the coefficient is not 1?

Preparation for a Test

- Review examples of each type of problem that could appear on the test.
- Review the homework problems your teacher assigned.
- Take a practice test.
Chapter 7  Polynomial Equations and Factoring

7.1–7.4  Quiz

Write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms.  (Section 7.1)

1. \(-8q^3\)  
2. \(9 + d^2 - 3d\)  
3. \(\frac{2}{3}m^4 - \frac{5}{6}m^6\)  
4. \(-1.3z + 3z^3 + 7.4z^2\)

Find the sum or difference.  (Section 7.1)

5. \((2x^2 + 5) + (-x^2 + 4)\)  
6. \((-3n^2 + n) - (2n^2 - 7)\)  
7. \((-p^2 + 4p) - (p^2 - 3p + 15)\)  
8. \((a^2 - 3ab + b^2) + (-a^2 + ab + b^3)\)

Find the product.  (Section 7.2 and Section 7.3)

9. \((w + 6)(w + 7)\)  
10. \((3 - 4d)(2d - 5)\)  
11. \((y + 9)(y^2 + 2y - 3)\)  
12. \((3z - 5)(3z + 5)\)  
13. \((t + 5)^2\)  
14. \((2q - 6)^2\)

Solve the equation.  (Section 7.4)

15. \(5x^2 - 15x = 0\)  
16. \((8 - g)(8 - g) = 0\)  
17. \((3p + 7)(3p - 7)(p + 8) = 0\)  
18. \(-3y(y - 8)(2y + 1) = 0\)

19. You are making a blanket with a fringe border of equal width on each side.  (Section 7.1 and Section 7.2)
   a. Write a polynomial that represents the perimeter of the blanket including the fringe.
   b. Write a polynomial that represents the area of the blanket including the fringe.
   c. Find the perimeter and the area of the blanket including the fringe when the width of the fringe is 4 inches.

20. You are saving money to buy an electric guitar. You deposit $1000 in an account that earns interest compounded annually. The expression \(1000(1 + r)^2\) represents the balance after 2 years, where \(r\) is the annual interest rate in decimal form.  (Section 7.3)
   a. Write the polynomial in standard form that represents the balance of your account after 2 years.
   b. The interest rate is 3%. What is the balance of your account after 2 years?
   c. The guitar costs $1100. Do you have enough money in your account after 3 years? Explain.

21. The front of a storage bunker can be modeled by \(y = -\frac{5}{216}(x - 72)(x + 72)\), where \(x\) and \(y\) are measured in inches. The \(x\)-axis represents the ground. Find the width of the bunker at ground level.  (Section 7.4)
7.5 Factoring $x^2 + bx + c$

**Essential Question** How can you use algebra tiles to factor the trinomial $x^2 + bx + c$ into the product of two binomials?

**Finding Binomial Factors**

Work with a partner. Use algebra tiles to write each polynomial as the product of two binomials. Check your answer by multiplying.

- **Sample** $x^2 + 5x + 6$
  
  **Step 1** Arrange algebra tiles that model $x^2 + 5x + 6$ into a rectangular array.
  
  **Step 2** Use additional algebra tiles to model the dimensions of the rectangle.
  
  **Step 3** Write the polynomial in factored form using the dimensions of the rectangle.
  
  \[
  \text{Area} = x^2 + 5x + 6 = (x + 2)(x + 3)
  \]

  - **a.** $x^2 - 3x + 2 = \boxed{}$
  - **b.** $x^2 + 5x + 4 = \boxed{}$
  
  - **c.** $x^2 - 7x + 12 = \boxed{}$
  - **d.** $x^2 + 7x + 12 = \boxed{}$

**Communicate Your Answer**

2. How can you use algebra tiles to factor the trinomial $x^2 + bx + c$ into the product of two binomials?

3. Describe a strategy for factoring the trinomial $x^2 + bx + c$ that does not use algebra tiles.
What You Will Learn

- Factor \( x^2 + bx + c \).
- Use factoring to solve real-life problems.

### Core Concept

#### Factoring \( x^2 + bx + c \) When \( c \) Is Positive

**Algebra**

\[ x^2 + bx + c = (x + p)(x + q) \]

When \( c \) is positive, \( p \) and \( q \) have the same sign as \( b \).

**Examples**

\[ x^2 + 6x + 5 = (x + 1)(x + 5) \]
\[ x^2 - 6x + 5 = (x - 1)(x - 5) \]

### Example 1

**Factoring \( x^2 + bx + c \) When \( b \) and \( c \) Are Positive**

Factor \( x^2 + 10x + 16 \).

**SOLUTION**

Notice that \( b = 10 \) and \( c = 16 \).

- Because \( c \) is positive, the factors \( p \) and \( q \) must have the same sign so that \( pq \) is positive.
- Because \( b \) is also positive, \( p \) and \( q \) must each be positive so that \( p + q \) is positive.

Find two positive integer factors of 16 whose sum is 10.

**Check**

Use the FOIL Method.

\[
(x + 2)(x + 8)
\]

\[
= x^2 + 8x + 2x + 16
\]

\[
= x^2 + 10x + 16 \checkmark
\]

<table>
<thead>
<tr>
<th>Factors of 16</th>
<th>Sum of factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 16</td>
<td>17</td>
</tr>
<tr>
<td>2, 8</td>
<td>10</td>
</tr>
<tr>
<td>4, 4</td>
<td>8</td>
</tr>
</tbody>
</table>

The values of \( p \) and \( q \) are 2 and 8.

So, \( x^2 + 10x + 16 = (x + 2)(x + 8) \).

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Factor the polynomial.

1. \( x^2 + 7x + 6 \)
2. \( x^2 + 9x + 8 \)
Factoring $x^2 + bx + c$ When $b$ Is Negative and $c$ Is Positive

Factor $x^2 - 8x + 12$.

**SOLUTION**

Notice that $b = -8$ and $c = 12$.

- Because $c$ is positive, the factors $p$ and $q$ must have the same sign so that $pq$ is positive.
- Because $b$ is negative, $p$ and $q$ must each be negative so that $p + q$ is negative.

Find two negative integer factors of 12 whose sum is $-8$.

<table>
<thead>
<tr>
<th>Factors of 12</th>
<th>-1, -12</th>
<th>-2, -6</th>
<th>-3, -4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum of factors</td>
<td>-13</td>
<td>-8</td>
<td>-7</td>
</tr>
</tbody>
</table>

The values of $p$ and $q$ are $-2$ and $-6$.

So, $x^2 - 8x + 12 = (x - 2)(x - 6)$.

**Core Concept**

Factoring $x^2 + bx + c$ When $c$ Is Negative

**Algebra**

$x^2 + bx + c = (x + p)(x + q)$ when $p + q = b$ and $pq = c$.

When $c$ is negative, $p$ and $q$ have different signs.

**Example**

$x^2 - 4x - 5 = (x + 1)(x - 5)$

**EXAMPLE 3** Factoring $x^2 + bx + c$ When $c$ Is Negative

Factor $x^2 + 4x - 21$.

**SOLUTION**

Notice that $b = 4$ and $c = -21$. Because $c$ is negative, the factors $p$ and $q$ must have different signs so that $pq$ is negative.

Find two integer factors of $-21$ whose sum is $4$.

<table>
<thead>
<tr>
<th>Factors of $-21$</th>
<th>-21, 1</th>
<th>-1, 21</th>
<th>-7, 3</th>
<th>-3, 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum of factors</td>
<td>-20</td>
<td>20</td>
<td>-4</td>
<td>4</td>
</tr>
</tbody>
</table>

The values of $p$ and $q$ are $-3$ and $7$.

So, $x^2 + 4x - 21 = (x - 3)(x + 7)$.

**Monitoring Progress**

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Factor the polynomial.

3. $w^2 - 4w + 3$  
4. $n^2 - 12n + 35$  
5. $x^2 - 14x + 24$  
6. $x^2 + 2x - 15$  
7. $y^2 + 13y - 30$  
8. $v^2 - v - 42$
Solving Real-Life Problems

**EXAMPLE 4** Solving a Real-Life Problem

A farmer plants a rectangular pumpkin patch in the northeast corner of a square plot of land. The area of the pumpkin patch is 600 square meters. What is the area of the square plot of land?

**SOLUTION**

1. **Understand the Problem** You are given the area of the pumpkin patch, the difference of the side length of the square plot and the length of the pumpkin patch, and the difference of the side length of the square plot and the width of the pumpkin patch.

2. **Make a Plan** The length of the pumpkin patch is \((s - 30)\) meters and the width is \((s - 40)\) meters. Write and solve an equation to find the side length \(s\). Then use the solution to find the area of the square plot of land.

3. **Solve the Problem** Use the equation for the area of a rectangle to write and solve an equation to find the side length \(s\) of the square plot of land.

   \[
   \begin{align*}
   600 &= (s - 30)(s - 40) \\
   600 &= s^2 - 70s + 1200 \\
   0 &= s^2 - 70s + 600 \\
   0 &= (s - 10)(s - 60) \\
   s - 10 &= 0 \text{ or } s - 60 &= 0 \\
   s &= 10 \text{ or } s &= 60
   \end{align*}
   \]

   So, the area of the square plot of land is \(60(60) = 3600\) square meters.

4. **Look Back** Use the diagram to check that you found the correct side length. Using \(s = 60\), the length of the pumpkin patch is \(60 - 30 = 30\) meters and the width is \(60 - 40 = 20\) meters. So, the area of the pumpkin patch is 600 square meters. This matches the given information and confirms the side length is 60 meters, which gives an area of 3600 square meters.

---

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9. **WHAT IF?** The area of the pumpkin patch is 200 square meters. What is the area of the square plot of land?

---

**Concept Summary**

Factoring \(x^2 + bx + c\) as \((x + p)(x + q)\)

The diagram shows the relationships between the signs of \(b\) and \(c\) and the signs of \(p\) and \(q\).

- \(c\) is positive. \(b\) is positive.
- \(c\) is positive. \(b\) is negative.
- \(c\) is negative.
- \(p\) and \(q\) are positive.
- \(p\) and \(q\) are negative.
- \(p\) and \(q\) have different signs.
In Exercises 3–8, factor the polynomial. (See Example 1.)

3. \( x^2 + 8x + 7 \)  
4. \( x^2 + 10z + 21 \)  
5. \( n^2 + 9n + 20 \)  
6. \( s^2 + 11s + 30 \)  
7. \( h^2 + 11h + 18 \)  
8. \( y^2 + 13y + 40 \)

In Exercises 9–14, factor the polynomial. (See Example 2.)

9. \( v^2 - 5v + 4 \)  
10. \( x^2 - 13x + 22 \)  
11. \( d^2 - 5d + 6 \)  
12. \( k^2 - 10k + 24 \)  
13. \( w^2 - 17w + 72 \)  
14. \( j^2 - 13j + 42 \)

In Exercises 15–24, factor the polynomial. (See Example 3.)

15. \( x^2 + 3x - 4 \)  
16. \( z^2 + 7z - 18 \)  
17. \( n^2 + 4n - 12 \)  
18. \( s^2 + 3s - 40 \)  
19. \( y^2 + 2y - 48 \)  
20. \( h^2 + 6h - 27 \)  
21. \( x^2 - x - 20 \)  
22. \( m^2 - 6m - 7 \)  
23. \( -6t - 16 + t^2 \)  
24. \( -7y + y^2 - 30 \)

25. MODELING WITH MATHEMATICS A projector displays an image on a wall. The area (in square feet) of the projection is represented by \( x^2 - 8x + 15 \).

a. Write a binomial that represents the height of the projection.

b. Find the perimeter of the projection when the height of the wall is 8 feet.

26. MODELING WITH MATHEMATICS A dentist’s office and parking lot are on a rectangular piece of land. The area (in square meters) of the land is represented by \( x^2 + x - 30 \).

a. Write a binomial that represents the width of the land.

b. Find the area of the land when the length of the dentist’s office is 20 meters.

ERROR ANALYSIS In Exercises 27 and 28, describe and correct the error in factoring the polynomial.

27. \[ x^2 + 14x + 48 = (x + 4)(x + 12) \]

28. \[ s^2 - 17s - 60 = (s - 5)(s - 12) \]

In Exercises 29–38, solve the equation.

29. \( m^2 + 3m + 2 = 0 \)  
30. \( n^2 - 9n + 18 = 0 \)  
31. \( x^2 + 5x - 14 = 0 \)  
32. \( v^2 + 11v - 26 = 0 \)  
33. \( t^2 + 15t = -36 \)  
34. \( n^2 - 5n = 24 \)  
35. \( a^2 + 5a - 20 = 30 \)  
36. \( y^2 - 2y - 8 = 7 \)  
37. \( m^2 + 10 = 15m - 34 \)  
38. \( b^2 + 5 = 8b - 10 \)

Section 7.5 Factoring \( x^2 + bx + c \)
39. **MODELING WITH MATHEMATICS** You trimmed a large square picture so that you could fit it into a frame. The area of the cut picture is 20 square inches. What is the area of the original picture? (See Example 4.)

![Image of a picture with dimensions 5 in. by 6 in.]

40. **MODELING WITH MATHEMATICS** A web browser is open on your computer screen.

![Image of a computer screen with dimensions (x - 7) in. by (x - 2) in.]

a. The area of the browser window is 24 square inches. Find the length of the browser window x.

b. The browser covers \( \frac{3}{13} \) of the screen. What are the dimensions of the screen?

41. **MAKING AN ARGUMENT** Your friend says there are six integer values of b for which the trinomial \( x^2 + bx - 12 \) has two binomial factors of the form \((x + p)\) and \((x + q)\). Is your friend correct? Explain.

42. **THOUGHT PROVOKING** Use algebra tiles to factor each polynomial modeled by the tiles. Show your work.

a. ![Image of algebra tiles with dimensions 1 by 1]

b. ![Image of algebra tiles with dimensions 1 by 1 and 2 by 2]

43. **MATHEMATICAL CONNECTIONS** In Exercises 43 and 44, find the dimensions of the polygon with the given area.

44. **MODELING WITH MATHEMATICS** You trimmed a large square picture so that you could fit it into a frame. The area of the cut picture is 20 square inches. What is the area of the original picture? (See Example 4.)

![Image of a picture with dimensions 5 in. by 6 in.]

45. **REASONING** Write an equation of the form \( x^2 + bx + c = 0 \) that has the solutions \( x = -4 \) and \( x = 6 \). Explain how you found your answer.

46. **HOW DO YOU SEE IT?** The graph of \( y = x^2 + x - 6 \) is shown.

![Graph of \( y = x^2 + x - 6 \)]

a. Explain how you can use the graph to factor the polynomial \( x^2 + x - 6 \).

b. Factor the polynomial.

47. **PROBLEM SOLVING** Road construction workers are paving the area shown.

![Image of a road with dimensions 18 m by 20 m]

a. Write an expression that represents the area being paved.

b. The area being paved is 280 square meters. Write and solve an equation to find the width of the road x.

48. **USING STRUCTURE** In Exercises 48–51, factor the polynomial.

49. \( r^2 + 7rs + 12s^2 \)

50. \( a^2 + 11ab - 26b^2 \)

51. \( x^2 - 2xy - 35y^2 \)

52. \( p - 9 = 0 \)

53. \( z + 12 = -5 \)

54. \( 6 = \frac{c}{-7} \)

55. \( 4k = 0 \)

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons.

52. \( p - 9 = 0 \)

53. \( z + 12 = -5 \)

54. \( 6 = \frac{c}{-7} \)

55. \( 4k = 0 \)
### Essential Question
How can you use algebra tiles to factor the trinomial $ax^2 + bx + c$ into the product of two binomials?

#### Exploration 1: Finding Binomial Factors

**Work with a partner.** Use algebra tiles to write each polynomial as the product of two binomials. Check your answer by multiplying.

**Sample**  $2x^2 + 5x + 2$

**Step 1** Arrange algebra tiles that model $2x^2 + 5x + 2$ into a rectangular array.

**Step 2** Use additional algebra tiles to model the dimensions of the rectangle.

**Step 3** Write the polynomial in factored form using the dimensions of the rectangle.

Area = $2x^2 + 5x + 2 = (x + 2)(2x + 1)$

**a.** $3x^2 + 5x + 2 = ($)

**b.** $4x^2 + 4x - 3 = ($)

**c.** $2x^2 - 11x + 5 = ($)

### Communicate Your Answer

2. How can you use algebra tiles to factor the trinomial $ax^2 + bx + c$ into the product of two binomials?

3. Is it possible to factor the trinomial $2x^2 + 2x + 1$? Explain your reasoning.
What You Will Learn

- Factor \( ax^2 + bx + c \).
- Use factoring to solve real-life problems.

Factoring \( ax^2 + bx + c \)

In Section 7.5, you factored polynomials of the form \( ax^2 + bx + c \), where \( a = 1 \). To factor polynomials of the form \( ax^2 + bx + c \), where \( a \neq 1 \), first look for the GCF of the terms of the polynomial and then factor further if possible.

**EXAMPLE 1** Factoring Out the GCF

Factor \( 5x^2 + 15x + 10 \).

**SOLUTION**

Notice that the GCF of the terms \( 5x^2, 15x, \) and \( 10 \) is \( 5 \).

\[
5x^2 + 15x + 10 = 5(x^2 + 3x + 2) \quad \text{Factor out GCF.}
\]

\[
= 5(x + 1)(x + 2) \quad \text{Factor } x^2 + 3x + 2.
\]

So, \( 5x^2 + 15x + 10 = 5(x + 1)(x + 2) \).

When there is no GCF, consider the possible factors of \( a \) and \( c \).

**EXAMPLE 2** Factoring \( ax^2 + bx + c \) When \( ac \) Is Positive

Factor each polynomial.

**a.** \( 4x^2 + 13x + 3 \)

**b.** \( 3x^2 - 7x + 2 \)

**SOLUTION**

**a.** There is no GCF, so you need to consider the possible factors of \( a \) and \( c \). Because \( b \) and \( c \) are both positive, the factors of \( c \) must be positive. Use a table to organize information about the factors of \( a \) and \( c \).

<table>
<thead>
<tr>
<th>Factors of 4</th>
<th>Factors of 3</th>
<th>Possible factorization</th>
<th>Middle term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 4</td>
<td>1, 3</td>
<td>((x + 1)(4x + 3))</td>
<td>(3x + 4x = 7x) X</td>
</tr>
<tr>
<td>1, 4</td>
<td>3, 1</td>
<td>((x + 3)(4x + 1))</td>
<td>(x + 12x = 13x) ✓</td>
</tr>
<tr>
<td>2, 2</td>
<td>1, 3</td>
<td>((2x + 1)(2x + 3))</td>
<td>(6x + 2x = 8x) X</td>
</tr>
</tbody>
</table>

So, \( 4x^2 + 13x + 3 = (x + 3)(4x + 1) \).

**b.** There is no GCF, so you need to consider the possible factors of \( a \) and \( c \). Because \( b \) is negative and \( c \) is positive, both factors of \( c \) must be negative. Use a table to organize information about the factors of \( a \) and \( c \).

<table>
<thead>
<tr>
<th>Factors of 3</th>
<th>Factors of 2</th>
<th>Possible factorization</th>
<th>Middle term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 3</td>
<td>-1, -2</td>
<td>((x - 1)(3x - 2))</td>
<td>(-2x - 3x = -5x) X</td>
</tr>
<tr>
<td>1, 3</td>
<td>-2, -1</td>
<td>((x - 2)(3x - 1))</td>
<td>(-x - 6x = -7x) ✓</td>
</tr>
</tbody>
</table>

So, \( 3x^2 - 7x + 2 = (x - 2)(3x - 1) \).
EXAMPLE 3  Factoring $ax^2 + bx + c$ When $ac$ Is Negative

Factor $2x^2 - 5x - 7$.

SOLUTION

There is no GCF, so you need to consider the possible factors of $a$ and $c$. Because $c$ is negative, the factors of $c$ must have different signs. Use a table to organize information about the factors of $a$ and $c$.

<table>
<thead>
<tr>
<th>Factors of 2</th>
<th>Factors of $-7$</th>
<th>Possible factorization</th>
<th>Middle term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>1, $-7$</td>
<td>$(x + 1)(2x - 7)$</td>
<td>$-7x + 2x = -5x$  ✔</td>
</tr>
<tr>
<td>1, 2</td>
<td>7, $-1$</td>
<td>$(x + 7)(2x - 1)$</td>
<td>$-x + 14x = 13x$  ❌</td>
</tr>
<tr>
<td>1, 2</td>
<td>$-1, 7$</td>
<td>$(x - 1)(2x + 7)$</td>
<td>$7x - 2x = 5x$  ❌</td>
</tr>
<tr>
<td>1, 2</td>
<td>$-7, 1$</td>
<td>$(x - 7)(2x + 1)$</td>
<td>$x - 14x = -13x$  ❌</td>
</tr>
</tbody>
</table>

So, $2x^2 - 5x - 7 = (x + 1)(2x - 7)$.

EXAMPLE 4  Factoring $ax^2 + bx + c$ When $a$ Is Negative

Factor $-4x^2 - 8x + 5$.

SOLUTION

Step 1  Factor $-1$ from each term of the trinomial.

$-4x^2 - 8x + 5 = -(4x^2 + 8x - 5)$

Step 2  Factor the trinomial $4x^2 + 8x - 5$. Because $c$ is negative, the factors of $c$ must have different signs. Use a table to organize information about the factors of $a$ and $c$.

<table>
<thead>
<tr>
<th>Factors of 4</th>
<th>Factors of $-5$</th>
<th>Possible factorization</th>
<th>Middle term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 4</td>
<td>1, $-5$</td>
<td>$(x + 1)(4x - 5)$</td>
<td>$-5x + 4x = -x$  ❌</td>
</tr>
<tr>
<td>1, 4</td>
<td>5, $-1$</td>
<td>$(x + 5)(4x - 1)$</td>
<td>$-x + 20x = 19x$  ✔</td>
</tr>
<tr>
<td>1, 4</td>
<td>$-1, 5$</td>
<td>$(x - 1)(4x + 5)$</td>
<td>$5x - 4x = x$  ✔</td>
</tr>
<tr>
<td>1, 4</td>
<td>$-5, 1$</td>
<td>$(x - 5)(4x + 1)$</td>
<td>$x - 20x = -19x$  ✔</td>
</tr>
<tr>
<td>2, 2</td>
<td>1, $-5$</td>
<td>$(2x + 1)(2x - 5)$</td>
<td>$-10x + 2x = -8x$  ✔</td>
</tr>
<tr>
<td>2, 2</td>
<td>$-1, 5$</td>
<td>$(2x - 1)(2x + 5)$</td>
<td>$10x - 2x = 8x$  ✔</td>
</tr>
</tbody>
</table>

So, $-4x^2 - 8x + 5 = -(2x - 1)(2x + 5)$.

Monitoring Progress  Help in English and Spanish at BigIdeasMath.com

Factor the polynomial.

1. $8x^2 - 56x + 48$  2. $14x^2 + 31x + 15$  3. $2x^2 - 7x + 5$
4. $3x^2 - 14x + 8$  5. $4x^2 - 19x - 5$  6. $6x^2 + x - 12$
7. $-2y^2 - 5y - 3$  8. $-5m^2 + 6m - 1$  9. $-3x^2 - x + 2$

Section 7.6  Factoring $ax^2 + bx + c$  393
Solving Real-Life Problems

Example 5  Solving a Real-Life Problem

The length of a rectangular game reserve is 1 mile longer than twice the width. The area of the reserve is 55 square miles. What is the width of the reserve?

SOLUTION

Use the formula for the area of a rectangle to write an equation for the area of the reserve. Let \( w \) represent the width. Then \( 2w + 1 \) represents the length. Solve for \( w \).

\[
\begin{align*}
  w(2w + 1) &= 55 & \text{Area of the reserve} \\
  2w^2 + w &= 55 & \text{Distributive Property} \\
  2w^2 + w - 55 &= 0 & \text{Subtract 55 from each side.}
\end{align*}
\]

Factor the left side of the equation. There is no GCF, so you need to consider the possible factors of \( a \) and \( c \). Because \( c \) is negative, the factors of \( c \) must have different signs. Use a table to organize information about the factors of \( a \) and \( c \).

<table>
<thead>
<tr>
<th>Factors of 2</th>
<th>Factors of (-55)</th>
<th>Possible factorization</th>
<th>Middle term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>1, −55</td>
<td>((w + 1)(2w - 55))</td>
<td>(-55w + 2w = -53w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>55, −1</td>
<td>((w + 55)(2w - 1))</td>
<td>(-w + 110w = 109w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>−1, 55</td>
<td>((w - 1)(2w + 55))</td>
<td>(55w - 2w = 53w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>−55, 1</td>
<td>((w - 55)(2w + 1))</td>
<td>(w - 110w = -109w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>5, −11</td>
<td>((w + 5)(2w - 11))</td>
<td>(-11w + 10w = -w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>11, −5</td>
<td>((w + 11)(2w - 5))</td>
<td>(-5w + 22w = 17w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>−5, 11</td>
<td>((w - 5)(2w + 11))</td>
<td>(11w - 10w = w)</td>
</tr>
<tr>
<td>1, 2</td>
<td>−11, 5</td>
<td>((w - 11)(2w + 5))</td>
<td>(5w - 22w = -17w)</td>
</tr>
</tbody>
</table>

So, you can rewrite \(2w^2 + w - 55\) as \((w - 5)(2w + 11)\). Write the equation with the left side factored and continue solving for \( w \).

\[
\begin{align*}
  (w - 5)(2w + 11) &= 0 & \text{Rewrite equation with left side factored.} \\
  w - 5 &= 0 & \text{or} & \quad 2w + 11 = 0 & \text{Zero-Product Property} \\
  w &= 5 & \text{or} & \quad w = -\frac{11}{2} & \text{Solve for } w.
\end{align*}
\]

A negative width does not make sense, so you should use the positive solution.

\[ \text{So, the width of the reserve is 5 miles.} \]

Monitoring Progress

10. WHAT IF? The area of the reserve is 136 square miles. How wide is the reserve?
7.6 Exercises

Vocabulary and Core Concept Check

1. REASONING What is the greatest common factor of the terms of $3y^2 - 21y + 36$?

2. WRITING Compare factoring $6x^2 - x - 2$ with factoring $x^2 - x - 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, factor the polynomial. (See Example 1.)

3. $3x^2 + 3x - 6$
4. $8m^2 + 8m - 48$
5. $4k^2 + 28k + 48$
6. $6y^2 - 24y + 18$
7. $7b^2 - 63b + 140$
8. $9r^2 - 36r - 45$

In Exercises 9–16, factor the polynomial. (See Examples 2 and 3.)

9. $3h^2 + 11h + 6$
10. $8m^2 + 30m + 7$
11. $6x^2 - 5x + 1$
12. $10u^2 - 31u + 15$
13. $3n^2 + 5n - 2$
14. $4z^2 + 4z - 3$
15. $8g^2 - 10g - 12$
16. $18v^2 - 15v - 18$

In Exercises 17–22, factor the polynomial. (See Example 4.)

17. $-3t^2 + 11t - 6$
18. $-7v^2 - 25v - 12$
19. $-4c^2 + 19c + 5$
20. $-8h^2 - 13h + 6$
21. $-15w^2 - w + 28$
22. $-22d^2 + 29d - 9$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in factoring the polynomial.

23. $2x^2 - 2x - 24 = 2(x^2 - x - 24) = 2(x - 6)(x + 4)$

24. $6x^2 - 7x - 3 = (3x - 3)(2x + 1)$

In Exercises 25–28, solve the equation.

25. $5x^2 - 5x - 30 = 0$
26. $2k^2 - 5k - 18 = 0$
27. $-12n^2 - 11n = -15$
28. $14h^2 - 2 = -3h$

In Exercises 29–32, find the $x$-coordinates of the points where the graph crosses the $x$-axis.

29. $y = 2x^2 - 3x - 35$
30. $y = 4x^2 + 11x - 3$
31. $y = -7x^2 - 2x + 5$
32. $y = -3x^2 + 14x + 5$

33. MODELING WITH MATHEMATICS The area (in square feet) of the school sign can be represented by $15x^2 - x - 2$.

a. Write an expression that represents the length of the sign.

b. Describe two ways to find the area of the sign when $x = 3$. 

Welcome to WESTFIELD MIDDLE SCHOOL

Home of the Conquers

(3x + 1) ft

Section 7.6 Factoring $ax^2 + bx + c$ 395
34. **MODELING WITH MATHEMATICS** The height \( h \) (in feet) above the water of a cliff diver is modeled by \( h = -16t^2 + 8t + 80 \), where \( t \) is the time (in seconds). How long is the diver in the air?

35. **MODELING WITH MATHEMATICS** The Parthenon in Athens, Greece, is an ancient structure that has a rectangular base. The length of the base of the Parthenon is 8 meters more than twice its width. The area of the base is about 2170 square meters. Find the length and width of the base. (See Example 5.)

36. **MODELING WITH MATHEMATICS** The length of a rectangular birthday party invitation is 1 inch less than twice its width. The area of the invitation is 15 square inches. Will the invitation fit in the envelope shown without being folded? Explain.

37. **OPEN-ENDED** Write a binomial whose terms have a GCF of \( 3x \).

38. **HOW DO YOU SEE IT?** Without factoring, determine which of the graphs represents the function \( g(x) = 21x^2 + 37x + 12 \) and which represents the function \( h(x) = 21x^2 - 37x + 12 \). Explain your reasoning.

39. **REASONING** When is it not possible to factor \( ax^2 + bx + c \), where \( a \neq 1 \)? Give an example.

40. **MAKING AN ARGUMENT** Your friend says that to solve the equation \( 5x^2 + x - 4 = 2 \), you should start by factoring the left side as \( (5x - 4)(x + 1) \). Is your friend correct? Explain.

41. **REASONING** For what values of \( t \) can \( 2t^3 + tx + 10 \) be written as the product of two binomials?

42. **THOUGHT PROVOKING** Use algebra tiles to factor each polynomial modeled by the tiles. Show your work.

43. **MATHEMATICAL CONNECTIONS** The length of a rectangle is 1 inch more than twice its width. The value of the area of the rectangle (in square inches) is 5 more than the value of the perimeter of the rectangle (in inches). Find the width.

44. **PROBLEM SOLVING** A rectangular swimming pool is bordered by a concrete patio. The width of the patio is the same on every side. The area of the surface of the pool is equal to the area of the patio. What is the width of the patio?

In Exercises 45–48, factor the polynomial.

45. \( 4k^2 + 7jk - 2j^2 \)

46. \( 6x^2 + 5xy - 4y^2 \)

47. \(-6a^2 + 19ab - 14b^2 \)

48. \( 18m^3 + 39m^2n - 15mn^2 \)

---

**Maintaining Mathematical Proficiency**

**Find the square root(s). (Skills Review Handbook)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm \sqrt{64} )</td>
<td>( \pm 8 )</td>
</tr>
<tr>
<td>( \sqrt{4} )</td>
<td>2</td>
</tr>
<tr>
<td>(-\sqrt{225} )</td>
<td>-15</td>
</tr>
<tr>
<td>( \pm \sqrt{81} )</td>
<td>( \pm 9 )</td>
</tr>
</tbody>
</table>

**Solve the system of linear equations by substitution. Check your solution. (Section 5.2)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 3 + 7x ) ( y - x = -3 )</td>
<td>( x = 0, y = 7 )</td>
</tr>
<tr>
<td>( 2x = y + 2 ) ( -x + 3y = 14 )</td>
<td>( x = 2, y = 6 )</td>
</tr>
<tr>
<td>( 5x - 2y = 14 ) ( -x + 3y = 14 )</td>
<td>( x = 0, y = 7 )</td>
</tr>
<tr>
<td>( -x - 8 = -y ) ( 9y - 12 + 3x = 0 )</td>
<td>( x = 1, y = 7 )</td>
</tr>
</tbody>
</table>
Section 7.7  Factoring Special Products

Essential Question  How can you recognize and factor special products?

**Exploration 1**  Factoring Special Products

*Work with a partner.* Use algebra tiles to write each polynomial as the product of two binomials. Check your answer by multiplying. State whether the product is a “special product” that you studied in Section 7.3.

a. $4x^2 - 1 = \hfill$  

b. $4x^2 - 4x + 1 = \hfill$

c. $4x^2 + 4x + 1 = \hfill$

d. $4x^2 - 6x + 2 = \hfill$

**Communication Your Answer**

3. How can you recognize and factor special products? Describe a strategy for recognizing which polynomials can be factored as special products.

4. Use the strategy you described in Question 3 to factor each polynomial.

   a. $25x^2 + 10x + 1$  
   b. $25x^2 - 10x + 1$  
   c. $25x^2 - 1$
7.7 Lesson

What You Will Learn

- Factor the difference of two squares.
- Factor perfect square trinomials.
- Use factoring to solve real-life problems.

Factoring the Difference of Two Squares

You can use special product patterns to factor polynomials.

Core Concept

**Difference of Two Squares Pattern**

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^2 - b^2 = (a + b)(a - b)$</td>
<td>$x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$</td>
</tr>
</tbody>
</table>

**EXAMPLE 1** Factoring the Difference of Two Squares

Factor (a) $x^2 - 25$ and (b) $4z^2 - 1$.

**SOLUTION**

a. $x^2 - 25 = x^2 - 5^2$
   
   $= (x + 5)(x - 5)$
   
   **Write as $a^2 - b^2$.**
   
   **Difference of two squares pattern**
   
   $\quad \Rightarrow$ So, $x^2 - 25 = (x + 5)(x - 5)$.

b. $4z^2 - 1 = (2z)^2 - 1^2$
   
   $= (2z + 1)(2z - 1)$
   
   **Write as $a^2 - b^2$.**
   
   **Difference of two squares pattern**
   
   $\quad \Rightarrow$ So, $4z^2 - 1 = (2z + 1)(2z - 1)$.

**EXAMPLE 2** Evaluating a Numerical Expression

Use a special product pattern to evaluate the expression $54^2 - 48^2$.

**SOLUTION**

Notice that $54^2 - 48^2$ is a difference of two squares. So, you can rewrite the expression in a form that it is easier to evaluate using the difference of two squares pattern.

$54^2 - 48^2 = (54 + 48)(54 - 48)$

$\quad = 102(6)$

$\quad = 612$

**Difference of two squares pattern**

$\quad \Rightarrow$ Simplify.

$\quad \Rightarrow$ Multiply.

$\quad \Rightarrow$ So, $54^2 - 48^2 = 612$.

Monitoring Progress

Factor the polynomial.

1. $x^2 - 36$
2. $100 - m^2$
3. $9n^2 - 16$
4. $16h^2 - 49$

Use a special product pattern to evaluate the expression.

5. $36^2 - 34^2$
6. $47^2 - 44^2$
7. $55^2 - 50^2$
8. $28^2 - 24^2$
Factoring Perfect Square Trinomials

**Core Concept**

**Perfect Square Trinomial Pattern**

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a^2 + 2ab + b^2 = (a + b)^2)</td>
<td>(x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2) = ((x + 3)^2)</td>
</tr>
<tr>
<td>(a^2 - 2ab + b^2 = (a - b)^2)</td>
<td>(x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2) = ((x - 3)^2)</td>
</tr>
</tbody>
</table>

**Example 3** Factoring Perfect Square Trinomials

Factor each polynomial.

a. \(n^2 + 8n + 16\)  
   b. \(4x^2 - 12x + 9\)

**SOLUTION**

a. \(n^2 + 8n + 16 = n^2 + 2(n)(4) + 4^2\)  
   Write as \(a^2 + 2ab + b^2\).  
   Perfect square trinomial pattern  
   \(= (n + 4)^2\)  
   So, \(n^2 + 8n + 16 = (n + 4)^2\).

b. \(4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + 3^2\)  
   Write as \(a^2 - 2ab + b^2\).  
   Perfect square trinomial pattern  
   \(= (2x - 3)^2\)  
   So, \(4x^2 - 12x + 9 = (2x - 3)^2\).

**Example 4** Solving a Polynomial Equation

Solve \(x^2 + \frac{2}{3}x + \frac{1}{9} = 0\).

**SOLUTION**

\[
x^2 + \frac{2}{3}x + \frac{1}{9} = 0
\]

\[
9x^2 + 6x + 1 = 0
\]

\[
(3x)^2 + 2(3x)(1) + 1^2 = 0
\]

\[
(3x + 1)^2 = 0
\]

\[
3x + 1 = 0
\]

\[
x = -\frac{1}{3}
\]

The solution is \(x = -\frac{1}{3}\).

**Looking for Structure**

Equations of the form \((x + a)^2 = 0\) always have repeated roots of \(x = -a\).

**Monitoring Progress**

Factor the polynomial.

9. \(m^2 - 2m + 1\)
10. \(d^2 - 10d + 25\)
11. \(9z^2 + 36z + 36\)

Solve the equation.

12. \(a^2 + 6a + 9 = 0\)
13. \(w^2 - \frac{2}{3}w + \frac{49}{36} = 0\)
14. \(n^2 - 81 = 0\)
Solving Real-Life Problems

**EXAMPLE 5  Modeling with Mathematics**

A bird picks up a golf ball and drops it while flying. The function represents the height $y$ (in feet) of the golf ball $t$ seconds after it is dropped. The ball hits the top of a 32-foot-tall pine tree. After how many seconds does the ball hit the tree?

**SOLUTION**

1. **Understand the Problem** You are given the height of the golf ball as a function of the amount of time after it is dropped and the height of the tree that the golf ball hits. You are asked to determine how many seconds it takes for the ball to hit the tree.

2. **Make a Plan** Use the function for the height of the golf ball. Substitute the height of the tree for $y$ and solve for the time $t$.

3. **Solve the Problem** Substitute 32 for $y$ and solve for $t$.

   \[ y = 81 - 16t^2 \]

   \[ 32 = 81 - 16t^2 \] \hspace{1cm} Write equation.

   \[ 0 = 49 - 16t^2 \] \hspace{1cm} Subtract 32 from each side.

   \[ 0 = 7^2 - (4t)^2 \] \hspace{1cm} Write as $a^2 - b^2$.

   \[ 0 = (7 + 4t)(7 - 4t) \] \hspace{1cm} Difference of two squares pattern

   \[ 7 + 4t = 0 \quad \text{or} \quad 7 - 4t = 0 \] \hspace{1cm} Zero-Product Property

   \[ t = \frac{-7}{4} \quad \text{or} \quad t = \frac{7}{4} \] \hspace{1cm} Solve for $t$.

   A negative time does not make sense in this situation.

   So, the golf ball hits the tree after \( \frac{7}{4} \), or 1.75 seconds.

4. **Look Back** Check your solution, as shown, by substituting $t = \frac{7}{4}$ into the equation

   \[ 32 = 81 - 16t^2 \]

   \[ 32 = 81 - 16\left(\frac{7}{4}\right)^2 \]

   \[ 32 = 81 - 16\left(\frac{49}{16}\right) \]

   \[ 32 = 81 - 49 \]

   \[ 32 = 32 \] \hspace{1cm} ✓

**Monitoring Progress**

15. **WHAT IF?** The golf ball does not hit the pine tree. After how many seconds does the ball hit the ground?
7.7 Exercises

Vocabulary and Core Concept Check

1. **REASONING** Can you use the perfect square trinomial pattern to factor \( y^2 + 16y + 64 \)? Explain.

2. **WHICH ONE DOESN’T BELONG?** Which polynomial does not belong with the other three? Explain your reasoning.

![Options](n^2 - 4, g^2 - 6g + 9, r^2 + 12r + 36, k^2 + 25)

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, factor the polynomial. (See Example 1.)

3. \( m^2 - 49 \)

4. \( z^2 - 81 \)

5. \( 64 - 81d^2 \)

6. \( 25 - 4x^2 \)

7. \( 225a^2 - 36b^2 \)

8. \( 16x^2 - 169y^2 \)

In Exercises 9–14, use a special product pattern to evaluate the expression. (See Example 2.)

9. \( 12^2 - 9^2 \)

10. \( 19^2 - 11^2 \)

11. \( 78^2 - 72^2 \)

12. \( 54^2 - 52^2 \)

13. \( 53^2 - 47^2 \)

14. \( 39^2 - 36^2 \)

In Exercises 15–22, factor the polynomial. (See Example 3.)

15. \( h^2 + 12h + 36 \)

16. \( p^2 + 30p + 225 \)

17. \( y^2 - 22y + 121 \)

18. \( x^2 - 4x + 4 \)

19. \( a^2 - 28a + 196 \)

20. \( m^2 + 24m + 144 \)

21. \( 25n^2 + 20n + 4 \)

22. \( 49a^2 - 14a + 1 \)

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in factoring the polynomial.

23. \( n^2 - 64 = n^2 - 8^2 \)

   \( = (n - 8)^2 \)

   \[ \text{CORRECTED: } n^2 - 16 \]

24. \( y^2 - 6y + 9 = y^2 - 2(3y) + 3^2 \)

   \( = (y - 3)(y + 3) \)

   \[ \text{CORRECTED: } (y - 3)^2 \]

In Exercises 23–40, solve the equation. (See Example 4.)

23. \( z^2 - 4 = 0 \)

24. \( 4x^2 = 49 \)

25. **MODELING WITH MATHEMATICS** The area (in square centimeters) of a square coaster can be represented by \( d^2 + 8d + 16 \).

   a. Write an expression that represents the perimeter of the coaster.

   b. Write an expression that represents the side length of the coaster.

26. **MODELING WITH MATHEMATICS** The polynomial \( A = x^2 - 30x + 225 \) represents the area (in square feet) of the square playground.

   a. Write a polynomial that represents the perimeter of the playground.

   b. Write an expression that represents the side length of the playground.

27. \( z^2 - 4 = 0 \)

28. \( 4x^2 = 49 \)

29. \( k^2 - 16k + 64 = 0 \)

30. \( x^2 + 20x + 100 = 0 \)

31. \( n^2 + 9 = 6n \)

32. \( y^2 = 12y - 36 \)

33. \( y^2 + \frac{1}{2}y = -\frac{1}{16} \)

34. \( -\frac{4}{3}x + \frac{4}{9} = -x^2 \)

In Exercises 35–40, factor the polynomial.

35. \( 3z^2 - 27 \)

36. \( 2m^2 - 50 \)

37. \( 4y^2 - 16y + 16 \)

38. \( 8x^2 + 80k + 200 \)

39. \( 50y^2 + 120y + 72 \)

40. \( 27m^2 - 36m + 12 \)

Section 7.7 Factoring Special Products
41. Modeling with Mathematics While standing on a ladder, you drop a paintbrush. The function represents the height \( y \) (in feet) of the paintbrush \( t \) seconds after it is dropped. After how many seconds does the paintbrush land on the ground? (See Example 5.)

\[
y = 25 - 16t^2
\]

42. Modeling with Mathematics The function represents the height \( y \) (in feet) of a grasshopper jumping straight up from the ground \( t \) seconds after the start of the jump. After how many seconds is the grasshopper 1 foot off the ground?

\[
y = -16t^2 + 8t
\]

43. Reasoning Tell whether the polynomial can be factored. If not, change the constant term so that the polynomial is a perfect square trinomial.
   a. \( w^2 + 18w + 84 \)
   b. \( y^2 - 10y + 23 \)

44. Thought provoking Use algebra tiles to factor each polynomial modeled by the tiles. Show your work.
   a. 
   b. 

45. Comparing methods Describe two methods you can use to simplify \((2x - 5)^2 - (x - 4)^2\). Which one would you use? Explain.

46. How do you see it? The figure shows a large square with an area of \( a^2 \) that contains a smaller square with an area of \( b^2 \).

a. Describe the regions that represent \( a^2 - b^2 \). How can you rearrange these regions to show that \( a^2 - b^2 = (a + b)(a - b) \)?
   b. How can you use the figure to show that \( (a - b)^2 = a^2 - 2ab + b^2 \)?

47. Problem solving You hang nine identical square picture frames on a wall.

a. Write a polynomial that represents the area of the picture frames, not including the pictures.
   b. The area in part (a) is 81 square inches. What is the side length of one of the picture frames? Explain your reasoning.

48. Mathematical connections The composite solid is made up of a cube and a rectangular prism.

a. Write a polynomial that represents the volume of the composite solid.
   b. The volume of the composite solid is equal to \( 25x \). What is the value of \( x \)? Explain your reasoning.

49. Write the prime factorization of the number. (Skills Review Handbook)
   49. 50  50. 44  51. 85  52. 96

50. Graph the inequality in a coordinate plane. (Section 5.6)
   53. \( y \leq 4x - 1 \)  54. \( y > -\frac{1}{2}x + 3 \)  55. \( 4y - 12 \geq 8x \)  56. \( 3y + 3 < x \)
7.8 Factoring Polynomials Completely

Essential Question  How can you factor a polynomial completely?

EXPLORATION 1  Writing a Product of Linear Factors

Work with a partner. Write the product represented by the algebra tiles. Then multiply to write the polynomial in standard form.

\[
\begin{align*}
\text{a. } & \quad (\textcolor{green}{+} \ +) (\textcolor{green}{+} \ +) (\textcolor{red}{-} \ -) \\
\text{b. } & \quad (\textcolor{green}{+} \ +) (\textcolor{green}{+} \ +) (\textcolor{red}{-} \ -) \\
\text{c. } & \quad (\textcolor{green}{+} \ + \ +) (\textcolor{green}{+} \ +) (\textcolor{red}{+} \ +) \\
\text{d. } & \quad (\textcolor{green}{+} \ +) (\textcolor{green}{-} \ -) (\textcolor{green}{+} \ +) \\
\text{e. } & \quad (\textcolor{green}{-} \ +) (\textcolor{green}{+} \ +) (\textcolor{red}{-} \ -) \\
\text{f. } & \quad (\textcolor{green}{-} \ -) (\textcolor{green}{+} \ +) (\textcolor{red}{-} \ -)
\end{align*}
\]

EXPLORATION 2  Matching Standard and Factored Forms

Work with a partner. Match the standard form of the polynomial with the equivalent factored form. Explain your strategy.

\[
\begin{align*}
\text{a. } & \quad x^3 + x^2 \quad & \quad \text{A. } & \quad x(x + 1)(x - 1) \\
\text{b. } & \quad x^3 - x \quad & \quad \text{B. } & \quad x(x - 1)^2 \\
\text{c. } & \quad x^3 + x^2 - 2x \quad & \quad \text{C. } & \quad x(x + 1)^2 \\
\text{d. } & \quad x^3 - 4x^2 + 4x \quad & \quad \text{D. } & \quad x(x + 2)(x - 1) \\
\text{e. } & \quad x^3 - 2x^2 - 3x \quad & \quad \text{E. } & \quad x(x - 1)(x - 2) \\
\text{f. } & \quad x^3 - 2x^2 + x \quad & \quad \text{F. } & \quad x(x + 2)(x - 2) \\
\text{g. } & \quad x^3 - 4x \quad & \quad \text{G. } & \quad x(x - 2)^2 \\
\text{h. } & \quad x^3 + 2x^2 \quad & \quad \text{H. } & \quad x(x + 2)^2 \\
\text{i. } & \quad x^3 - x^2 \quad & \quad \text{I. } & \quad x^2(x - 1) \\
\text{j. } & \quad x^3 - 3x^2 + 2x \quad & \quad \text{J. } & \quad x^2(x + 1) \\
\text{k. } & \quad x^3 + 2x^2 - 3x \quad & \quad \text{K. } & \quad x^2(x - 2) \\
\text{l. } & \quad x^3 - 4x^2 + 3x \quad & \quad \text{L. } & \quad x^2(x + 2) \\
\text{m. } & \quad x^3 - 2x^2 \quad & \quad \text{M. } & \quad x(x + 3)(x - 1) \\
\text{n. } & \quad x^3 + 4x^2 + 4x \quad & \quad \text{N. } & \quad x(x + 1)(x - 3) \\
\text{o. } & \quad x^3 + 2x^2 + x \quad & \quad \text{O. } & \quad x(x - 1)(x - 3)
\end{align*}
\]

Communicate Your Answer

3. How can you factor a polynomial completely?

4. Use your answer to Question 3 to factor each polynomial completely.

\[
\begin{align*}
\text{a. } & \quad x^3 + 4x^2 + 3x \quad & \quad \text{b. } & \quad x^3 - 6x^2 + 9x \quad & \quad \text{c. } & \quad x^3 + 6x^2 + 9x
\end{align*}
\]
What You Will Learn

- Factor polynomials by grouping.
- Factor polynomials completely.
- Use factoring to solve real-life problems.

Factoring Polynomials by Grouping

You have used the Distributive Property to factor out a greatest common monomial from a polynomial. Sometimes, you can factor out a common binomial. You may be able to use the Distributive Property to factor polynomials with four terms, as described below.

**Factoring by Grouping**

To factor a polynomial with four terms, group the terms into pairs. Factor the GCF out of each pair of terms. Look for and factor out the common binomial factor. This process is called **factoring by grouping**.

**Example 1**

Factor each polynomial by grouping.

a. \( x^3 + 3x^2 + 2x + 6 \)  
   b. \( x^2 + y + x + xy \)

**SOLUTION**

a. \( x^3 + 3x^2 + 2x + 6 = (x^3 + 3x^2) + (2x + 6) \)
   \[ = x^2(x + 3) + 2(x + 3) \]
   \[ = (x + 3)(x^2 + 2) \]
   
   **Common binomial factor is** \( x + 3 \).  
   
   So, \( x^3 + 3x^2 + 2x + 6 = (x + 3)(x^2 + 2) \).

b. \( x^2 + y + x + xy = x^2 + x + xy + y \)
   \[ = (x^2 + x) + (xy + y) \]
   \[ = x(x + 1) + y(x + 1) \]
   \[ = (x + 1)(x + y) \]

   **Common binomial factor is** \( x + 1 \).  
   
   So, \( x^2 + y + x + xy = (x + 1)(x + y) \).

**Monitoring Progress**

Factor the polynomial by grouping.

1. \( a^3 + 3a^2 + a + 3 \)  
2. \( y^2 + 2x + yx + 2y \)

Factoring Polynomials Completely

You have seen that the polynomial \( x^2 - 1 \) can be factored as \((x + 1)(x - 1)\). This polynomial is factorable. Notice that the polynomial \( x^2 + 1 \) cannot be written as the product of polynomials with integer coefficients. This polynomial is unfactorable.

A factorable polynomial with integer coefficients is **factored completely** when it is written as a product of unfactorable polynomials with integer coefficients.
### Concept Summary

**Guidelines for Factoring Polynomials Completely**

To factor a polynomial completely, you should try each of these steps:

1. Factor out the greatest common monomial factor.
   
   \[3x^2 + 6x = 3x(x + 2)\]

2. Look for a difference of two squares or a perfect square trinomial.
   
   \[x^2 + 4x + 4 = (x + 2)^2\]

3. Factor a trinomial of the form \(ax^2 + bx + c\) into a product of binomial factors.
   
   \[3x^2 - 5x - 2 = (3x + 1)(x - 2)\]

4. Factor a polynomial with four terms by grouping.
   
   \[x^3 + x - 4x^2 - 4 = (x^2 + 1)(x - 4)\]

### Example 2

**Factoring Completely**

Factor (a) \(3x^3 + 6x^2 - 18x\) and (b) \(7x^4 - 28x^2\).

**SOLUTION**

**a.** \(3x^3 + 6x^2 - 18x = 3x(x^2 + 2x - 6)\)

\[x^2 + 2x - 6\] is unfactorable, so the polynomial is factored completely.

So, \(3x^3 + 6x^2 - 18x = 3x(x^2 + 2x - 6)\).

**b.** \(7x^4 - 28x^2 = 7x^2(x^2 - 4)\)

\[= 7x^2(x^2 - 2^2)\]

\[= 7x^2(x + 2)(x - 2)\]

So, \(7x^4 - 28x^2 = 7x^2(x + 2)(x - 2)\).

### Example 3

**Solving an Equation by Factoring Completely**

Solve \(2x^3 + 8x^2 = 10x\).

**SOLUTION**

\[2x^3 + 8x^2 = 10x\]  \(\text{Original equation}\)

\[2x^3 + 8x^2 - 10x = 0\]  \(\text{Subtract 10x from each side.}\)

\[2x(x^2 + 4x - 5) = 0\]  \(\text{Factor out 2x.}\)

\[2x(x + 5)(x - 1) = 0\]  \(\text{Factor } x^2 + 4x - 5.\)

\[2x = 0 \quad or \quad x + 5 = 0 \quad or \quad x - 1 = 0\]  \(\text{Zero-Product Property}\)

\[x = 0 \quad or \quad x = -5 \quad or \quad x = 1\]  \(\text{Solve for } x.\)

The roots are \(x = -5, x = 0,\) and \(x = 1.\)

### Monitoring Progress

**Help in English and Spanish at BigIdeasMath.com**

Factor the polynomial completely.

3. \(3x^3 - 12x\)

4. \(2y^3 - 12y^2 + 18y\)

5. \(m^3 - 2m^2 - 8m\)

Solve the equation.

6. \(w^3 - 8w^2 + 16w = 0\)

7. \(x^3 - 25x = 0\)

8. \(c^3 - 7c^2 + 12c = 0\)
Solving Real-Life Problems

**EXAMPLE 4** Modeling with Mathematics

A terrarium in the shape of a rectangular prism has a volume of 4608 cubic inches. Its length is more than 10 inches. The dimensions of the terrarium in terms of its width are shown. Find the length, width, and height of the terrarium.

**SOLUTION**

1. **Understand the Problem** You are given the volume of a terrarium in the shape of a rectangular prism and a description of the length. The dimensions are written in terms of its width. You are asked to find the length, width, and height of the terrarium.

2. **Make a Plan** Use the formula for the volume of a rectangular prism to write and solve an equation for the width of the terrarium. Then substitute that value in the expressions for the length and height of the terrarium.

3. **Solve the Problem**

   Volume = length \( \times \) width \( \times \) height  
   
   \[ V = (36 - w)(w)(w + 4) \]  
   
   Write equation.

   \[ 4608 = (36 - w)w(w + 4) \]  
   
   Multiply.

   \[ 0 = 32w^2 + 144w - w^3 - 4608 \]  
   
   Subtract 4608 from each side.

   \[ 0 = (-w^3 + 32w^2) + (144w - 4608) \]  
   
   Group terms with common factors.

   \[ 0 = -w^2(w - 32) + 144(w - 32) \]  
   
   Factor out GCF of each pair of terms.

   \[ 0 = (w - 32)(-w^2 + 144) \]  
   
   Factor out \((w - 32)\).

   \[ 0 = -1(w - 32)(w^2 - 144) \]  
   
   Factor \(-1\) from \(-w^2 + 144\).

   \[ 0 = -1(w - 32)(w - 12)(w + 12) \]  
   
   Difference of two squares pattern

   \[ w - 32 = 0 \] \text{ or } \[ w - 12 = 0 \] \text{ or } \[ w + 12 = 0 \]  
   
   Zero-Product Property

   \[ w = 32 \] \text{ or } \[ w = 12 \] \text{ or } \[ w = -12 \]  
   
   Solve for \(w\).

   Disregard \(w = -12\) because a negative width does not make sense. You know that the length is more than 10 inches. Test the solutions of the equation, 12 and 32, in the expression for the length.

   length = 36 - w = 36 - 12 = 24 \( \checkmark \) \text{ or } length = 36 - w = 36 - 32 = 4 \( \times \)

   The solution 12 gives a length of 24 inches, so 12 is the correct value of \(w\).

   Use \(w = 12\) to find the height, as shown.

   height = \(w + 4\) = 12 + 4 = 16

   \( \rightarrow \) The width is 12 inches, the length is 24 inches, and the height is 16 inches.

4. **Look Back** Check your solution. Substitute the values for the length, width, and height when the width is 12 inches into the formula for volume. The volume of the terrarium should be 4608 cubic inches.

**Monitoring Progress**

9. A box in the shape of a rectangular prism has a volume of 72 cubic feet. The box has a length of \(x\) feet, a width of \((x - 1)\) feet, and a height of \((x + 9)\) feet. Find the dimensions of the box.
In Exercises 23–28, solve the equation.  
(See Example 2.)

23. \(5n^3 - 30n^2 + 40n = 0\)  
24. \(k^4 - 100k^2 = 0\)

25. \(x^3 + x^2 = 4x + 4\)  
26. \(2t^5 + 2t^4 - 144t^3 = 0\)

27. \(12x - 3x^3 = 0\)  
28. \(4y^3 - 7y^2 + 28 = 16y\)

In Exercises 29–32, find the \(x\)-coordinates of the points where the graph crosses the \(x\)-axis.

29. \(y = x^3 - 81x\)  
30. \(y = x^4 - 24x^3 - 45x^2\)

In Exercises 33–34, describe and correct the error in factoring the polynomial completely.

33. \(a^3 + 8a^2 - 6a - 48 = a^2(a + 8) + 6(a + 8)\)  
   \(= (a + 8)(a^2 + 6)\)

34. \(x^3 - 6x^2 - 9x + 54 = x^2(x - 6) - 9(x - 6)\)  
   \(= (x - 6)(x^2 - 9)\)

35. **MODELING WITH MATHEMATICS**

   You are building a birdhouse in the shape of a rectangular prism that has a volume of 128 cubic inches. The dimensions of the birdhouse in terms of \(w\) and \(h\) are shown.  
   (See Example 4.)

   a. Write a polynomial that represents the volume of the birdhouse.

   b. What are the dimensions of the birdhouse?
36. **MODELING WITH MATHEMATICS** A gift bag shaped like a rectangular prism has a volume of 1152 cubic inches. The dimensions of the gift bag in terms of its width are shown. The height is greater than the width. What are the dimensions of the gift bag?

(2w + 4) in.

w in.

In Exercises 37–40, factor the polynomial completely.

37. $x^3 + 2x^2y - x - 2y$

38. $8b^3 - 4b^2a - 18b + 9a$

39. $4x^2 - s + 12st - 3t$

40. $6m^3 - 12mn + m^2n - 2n^2$

41. **WRITING** Is it possible to find three real solutions of the equation $x^3 + 2x^2 + 3x + 6 = 0$? Explain your reasoning.

42. **HOW DO YOU SEE IT?** How can you use the factored form of the polynomial $x^4 - 2x^3 - 9x^2 + 18x = x(x - 3)(x + 3)(x - 2)$ to find the $x$-intercepts of the graph of the function?

43. **OPEN-ENDED** Write a polynomial of degree 3 that satisfies each of the given conditions.

   a. is not factorable
   b. can be factored by grouping

44. **MAKING AN ARGUMENT** Your friend says that if a trinomial cannot be factored as the product of two binomials, then the trinomial is factorable completely. Is your friend correct? Explain.

45. **PROBLEM SOLVING** The volume (in cubic feet) of a room in the shape of a rectangular prism is represented by $12z^3 - 27z$. Find expressions that could represent the dimensions of the room.

46. **MATHEMATICAL CONNECTIONS** The width of a box in the shape of a rectangular prism is 4 inches more than the height $h$. The length is the difference of 9 inches and the height.

   a. Write a polynomial that represents the volume of the box in terms of its height (in inches).
   b. The volume of the box is 180 cubic inches. What are the possible dimensions of the box?
   c. Which dimensions result in a box with the least possible surface area? Explain your reasoning.

47. **MATHEMATICAL CONNECTIONS** The volume of a cylinder is given by $V = \pi r^2h$, where $r$ is the radius of the base of the cylinder and $h$ is the height of the cylinder. Find the dimensions of the cylinder.

48. **THOUGHT PROVOKING** Factor the polynomial $x^5 - x^4 - 5x^3 + 5x^2 + 4x - 4$ completely.

49. **REASONING** Find a value for $w$ so that the equation has (a) two solutions and (b) three solutions. Explain your reasoning.

   $$5x^3 + wx^2 + 80x = 0$$

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations by graphing. **(Section 5.1)**

50. $y = x - 4$

   $y = -2x + 2$

51. $y = \frac{1}{2}x + 2$

   $y = 3x - 3$

52. $5x - y = 12$

   $\frac{1}{3}x + y = 9$

53. $x = 3y$

   $y - 10 = 2x$

Graph the function. Describe the domain and range. **(Section 6.3)**

54. $f(x) = 5^x$

55. $y = 9\left(\frac{1}{3}\right)^x$

56. $y = -3(0.5)^x$

57. $f(x) = -3(4)^x$
7.5–7.8 What Did You Learn?

Core Vocabulary

- factoring by grouping, p. 404
- factored completely, p. 404

Core Concepts

Section 7.5
- Factoring $x^2 + bx + c$ When $c$ Is Positive, p. 386
- Factoring $x^2 + bx + c$ When $c$ Is Negative, p. 387

Section 7.6
- Factoring $ax^2 + bx + c$ When $ac$ Is Positive, p. 392
- Factoring $ax^2 + bx + c$ When $ac$ Is Negative, p. 393

Section 7.7
- Difference of Two Squares Pattern, p. 398
- Perfect Square Trinomial Pattern, p. 399

Section 7.8
- Factoring by Grouping, p. 404
- Factoring Polynomials Completely, p. 404

Mathematical Practices

1. How are the solutions of Exercise 29 on page 389 related to the graph of $y = m^2 + 3m + 2$?

2. The equation in part (b) of Exercise 47 on page 390 has two solutions. Are both solutions of the equation reasonable in the context of the problem? Explain your reasoning.

Performance Task

The View Matters

The way an equation or expression is written can help you interpret and solve problems. Which representation would you rather have when trying to solve for specific information? Why?

To explore the answers to these questions and more, go to BigIdeasMath.com.
### 7.1 Adding and Subtracting Polynomials (pp. 357–364)

Find \((2x^3 + 6x^2 - x) - (-3x^3 - 2x^2 - 9x)\).

\[
(2x^3 + 6x^2 - x) - (-3x^3 - 2x^2 - 9x) = (2x^3 + 6x^2 - x) + (3x^3 + 2x^2 + 9x)
\]

\[
= 5x^3 + 8x^2 + 8x
\]

Write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms.

1. \(6 + 2x^2\)
2. \(-3p^3 + 5p^5 - 4\)
3. \(9x^3 - 6x^2 + 13x^5\)
4. \(-12y + 8y^3\)

Find the sum or difference.

5. \((3a + 7) + (a - 1)\)
6. \((x^2 + 6x - 5) + (2x^2 + 15)\)
7. \((-y^2 + y + 2) - (y^2 - 5y - 2)\)
8. \((p + 7) - (6p^2 + 13p)\)

### 7.2 Multiplying Polynomials (pp. 365–370)

Find \((x + 7)(x - 9)\).

\[
(x + 7)(x - 9) = x(x - 9) + 7(x - 9)
\]

\[
= x(x) + x(-9) + 7(x) + 7(-9)
\]

\[
= x^2 - 9x + 7x - 63
\]

Find the product.

9. \((x + 6)(x - 4)\)
10. \((y - 5)(3y + 8)\)
11. \((x + 4)(x^2 + 7x)\)
12. \((-3y + 1)(4y^2 - y - 7)\)

### 7.3 Special Products of Polynomials (pp. 371–376)

Find each product.

a. \((6x + 4y)^2\)

\[
(6x + 4y)^2 = (6x)^2 + 2(6x)(4y) + (4y)^2
\]

\[
= 36x^2 + 48xy + 16y^2
\]

b. \((2x + 3y)(2x - 3y)\)

\[
(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2
\]

\[
= 4x^2 - 9y^2
\]

Find the product.

13. \((x + 9)(x - 9)\)
14. \((2y + 4)(2y - 4)\)
15. \((p + 4)^2\)
16. \((-1 + 2d)^2\)
7.4 Solving Polynomial Equations in Factored Form (pp. 377–382)

Solve \((x + 6)(x - 8) = 0\).

\[(x + 6)(x - 8) = 0\]  
Write equation.

\[x + 6 = 0 \quad \text{or} \quad x - 8 = 0\]  
Zero-Product Property

\[x = -6 \quad \text{or} \quad x = 8\]  
Solve for \(x\).

17. \(x^2 + 5x = 0\)  
18. \((z + 3)(z - 7) = 0\)  
19. \((b + 13)^2 = 0\)  
20. \(2y(y - 9)(y + 4) = 0\)

7.5 Factoring \(x^2 + bx + c\) (pp. 385–390)

Factor \(x^2 + 6x - 27\).

Notice that \(b = 6\) and \(c = -27\). Because \(c\) is negative, the factors \(p\) and \(q\) must have different signs so that \(pq\) is negative.

Find two integer factors of \(-27\) whose sum is 6.

| Factors of \(-27\) | \(-27, 1\) | \(-1, 27\) | \(-9, 3\) | \(-3, 9\) |
| Sum of factors | \(-26\) | \(26\) | \(-6\) | \(6\) |

The values of \(p\) and \(q\) are \(-3\) and 9.

So, \(x^2 + 6x - 27 = (x - 3)(x + 9)\).

Factor the polynomial.

21. \(p^2 + 2p - 35\)  
22. \(b^2 + 18b + 80\)  
23. \(z^2 - 4z - 21\)  
24. \(x^2 - 11x + 28\)

7.6 Factoring \(ax^2 + bx + c\) (pp. 391–396)

Factor \(5x^2 + 36x + 7\).

There is no GCF, so you need to consider the possible factors of \(a\) and \(c\). Because \(b\) and \(c\) are both positive, the factors of \(c\) must be positive. Use a table to organize information about the factors of \(a\) and \(c\).

<table>
<thead>
<tr>
<th>Factors of 5</th>
<th>Factors of 7</th>
<th>Possible factorization</th>
<th>Middle term</th>
</tr>
</thead>
</table>
| 1, 5         | 1, 7         | \((x + 1)(5x + 7)\)   | \(7x + 5x = 12x\)  
\(\times\) |
| 1, 5         | 7, 1         | \((x + 7)(5x + 1)\)   | \(x + 35x = 36x\)  
\(\checkmark\) |

So, \(5x^2 + 36x + 7 = (x + 7)(5x + 1)\).

Factor the polynomial.

25. \(3x^2 + 16x - 12\)  
26. \(-5y^2 - 22y - 8\)  
27. \(6x^2 + 17x + 7\)  
28. \(-2y^2 + 7y - 6\)  
29. \(3z^2 + 26z - 9\)  
30. \(10a^2 - 13a - 3\)
7.7 Factoring Special Products (pp. 397–402)

Factor each polynomial.

a. \(x^2 - 16\)
   \[x^2 - 16 = x^2 - 4^2\] Write as \(a^2 - b^2\).
   \[= (x + 4)(x - 4)\] Difference of two squares pattern

b. \(25x^2 - 30x + 9\)
   \[25x^2 - 30x + 9 = (5x)^2 - 2(5x)(3) + 3^2\] Write as \(a^2 - 2ab + b^2\).
   \[= (5x - 3)^2\] Perfect square trinomial pattern

Factor the polynomial.

31. \(x^2 - 9\)  
32. \(y^2 - 100\)  
33. \(z^2 - 6z + 9\)  
34. \(m^2 + 16m + 64\)

7.8 Factoring Polynomials Completely (pp. 403–408)

Factor each polynomial completely.

a. \(x^3 + 4x^2 - 3x - 12\)
   \[x^3 + 4x^2 - 3x - 12 = (x^3 + 4x^2) + (-3x - 12)\] Group terms with common factors.
   \[= x^2(x + 4) + (-3)(x + 4)\] Factor out GCF of each pair of terms.
   \[= (x + 4)(x^2 - 3)\] Factor out \((x + 4)\).

b. \(2x^4 - 8x^2\)
   \[2x^4 - 8x^2 = 2x^2(x^2 - 4)\] Factor out \(2x^2\).
   \[= 2x^2(x^2 - 2^2)\] Write as \(a^2 - b^2\).
   \[= 2x^2(x + 2)(x - 2)\] Difference of two squares pattern

c. \(2x^3 + 18x^2 - 72x\)
   \[2x^3 + 18x^2 - 72x = 2x(x^2 + 9x - 36)\] Factor out \(2x\).
   \[= 2x(x + 12)(x - 3)\] Factor \(x^2 + 9x - 36\).

Factor the polynomial completely.

35. \(n^3 - 9n\)  
36. \(x^2 - 3x + 4ax - 12a\)  
37. \(2x^4 + 2x^3 - 20x^2\)

Solve the equation.

38. \(3x^3 - 9x^2 - 54x = 0\)  
39. \(16x^2 - 36 = 0\)  
40. \(z^3 + 3z^2 - 25z - 75 = 0\)

41. A box in the shape of a rectangular prism has a volume of 96 cubic feet. The box has a length of \((x + 8)\) feet, a width of \(x\) feet, and a height of \((x - 2)\) feet. Find the dimensions of the box.
Find the sum or difference. Then identify the degree of the sum or difference and classify it by the number of terms.

1. \((-2p + 4) - (p^2 - 6p + 8)\)  
2. \((9c^6 - 5b^4) - (4c^6 - 5b^4)\)  
3. \((4s^4 + 2st + t) + (2s^4 - 2st - 4t)\)

Find the product.

4. \((h - 5)(h - 8)\)  
5. \((2w - 3)(3w + 5)\)  
6. \((z + 11)(z - 11)\)

7. Explain how you can determine whether a polynomial is a perfect square trinomial.

8. Is 18 a polynomial? Explain your reasoning.

Factor the polynomial completely.

9. \(x^2 - 15x + 50\)  
10. \(h^3 + 2h^2 - 9h - 18\)  
11. \(-5k^2 - 22k + 15\)

Solve the equation.

12. \((n - 1)(n + 6)(n + 5) = 0\)  
13. \(d^2 + 14d + 49 = 0\)  
14. \(6x^4 + 8x^2 = 26x^3\)

15. The expression \(\pi(r - 3)^2\) represents the area covered by the hour hand on a clock in one rotation, where \(r\) is the radius of the entire clock. Write a polynomial in standard form that represents the area covered by the hour hand in one rotation.

16. A magician’s stage has a trapdoor.
   a. The total area (in square feet) of the stage can be represented by \(x^2 + 27x + 176\). Write an expression for the width of the stage.
   b. Write an expression for the perimeter of the stage.
   c. The area of the trapdoor is 10 square feet. Find the value of \(x\).
   d. The magician wishes to have the area of the stage be at least 20 times the area of the trapdoor. Does this stage satisfy his requirement? Explain.

17. Write a polynomial equation in factored form that has three positive roots.

18. You are jumping on a trampoline. For one jump, your height \(y\) (in feet) above the trampoline after \(t\) seconds can be represented by \(y = -16t^2 + 24t\). How many seconds are you in the air?

19. A cardboard box in the shape of a rectangular prism has the dimensions shown.
   a. Write a polynomial that represents the volume of the box.
   b. The volume of the box is 60 cubic inches. What are the length, width, and height of the box?
1. Classify each polynomial by the number of terms. Then order the polynomials by degree from least to greatest.

   a. \(-4x^3\)  
   b. \(6y - 3y^5\)  
   c. \(c^2 + 2 + c\)  
   d. \(-10d^4 + 7d^2\)  
   e. \(-5z^{11} + 8z^{12}\)  
   f. \(3b^6 - 12b^8 + 4b^4\)

2. Which exponential function is increasing the fastest over the interval \(x = 0\) to \(x = 2\)?

   \(\text{A} \quad f(x) = 4(2.5)^x\)
   \(\text{B} \quad \begin{array}{c|c|c|c|c|c} x & 0 & 1 & 2 & 3 & 4 \\ \hline g(x) & 8 & 12 & 18 & 27 & 40.5 \end{array}\)
   \(\text{C} \quad \text{A graph showing an exponential function increasing rapidly.}\)
   \(\text{D} \quad \text{An exponential function models a relationship in which the dependent variable is multiplied by 6 for every 1 unit the independent variable increases. The value of the function at 0 is 2.}\)

3. Find all solutions of the equation \(x^3 + 6x^2 - 4x = 24\).

   \(-6 \quad -4 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 4 \quad 6 \quad 24\)

4. The table shows the distances you travel over a 6-hour period. Create an equation that models the distance traveled as a function of the number of hours.

   \begin{array}{c|c} 
   \text{Hours, } x & \text{Distance (miles), } y \\
   \hline 
   1 & 62 \\
   2 & 123 \\
   3 & 184 \\
   4 & 245 \\
   5 & 306 \\
   6 & 367 \\
   \end{array}

5. Consider the equation \(y = -\frac{1}{3}x + 2\).

   a. Graph the equation in a coordinate plane.
   b. Does the equation represent a linear or nonlinear function?
   c. Is the domain discrete or continuous?
6. Which expressions are equivalent to $-2x + 15x^2 - 8$?

- $15x^2 - 2x - 8$
- $(5x - 4)(3x + 2)$
- $(3x - 2)(5x - 4)$
- $(5x + 4)(3x + 2)$
- $15x^2 + 2x - 8$
- $(3x + 2)(5x - 4)$

7. The graph shows the function $f(x) = 2(3)^x$.

![Graph of the function $f(x) = 2(3)^x$.]

a. Is the function increasing or decreasing for increasing values of $x$?
b. Identify any $x$- and $y$-intercepts.

8. Which polynomial represents the product of $2x - 4$ and $x^2 + 6x - 2$?

- A $2x^3 + 8x^2 - 4x + 8$
- B $2x^3 + 8x^2 - 28x + 8$
- C $2x^3 + 8$
- D $2x^3 - 24x - 2$

9. You are playing miniature golf on the hole shown.

a. Write a polynomial that represents the area of the golf hole.
b. Write a polynomial that represents the perimeter of the golf hole.
c. Find the perimeter of the golf hole when the area is 216 square feet.