6.2 Bisectors of Triangles

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Learning Target Use bisectors of triangles.

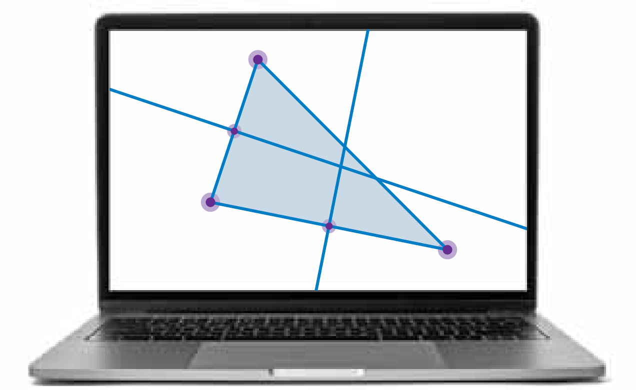
Success Criteria
- I can find the circumcenter and incenter of a triangle.
- I can circumscribe a circle about a triangle.
- I can inscribe a circle within a triangle.
- I can use points of concurrency to solve real-life problems.

EXPLORE IT! Analyzing Bisectors of Triangles

TENNESSEE MATH STANDARDS
G.CO.C.9,
G.CO.D.12

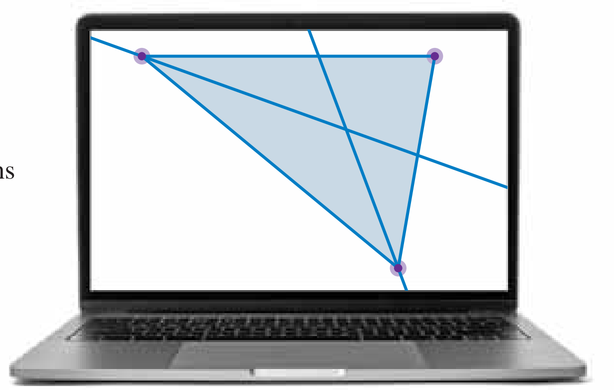
Work with a partner.

a. Use technology to draw any triangle and the perpendicular bisectors of all three sides of the triangle. What do you notice? What happens when you move the vertices of the triangle?



b. Draw the circle with its center at the intersection of the perpendicular bisectors that passes through a vertex of the triangle. What do you notice? What does this mean?

c. Use technology to draw a different triangle and the angle bisectors of all three angles of the triangle. What do you notice? What happens when you move the vertices of the triangle?



d. Find the distance r between the intersection of the angle bisectors and one of the sides of the triangle. Draw the circle with its center at the intersection of the angle bisectors and radius r. What do you notice? What does this mean?

e. What conjectures can you make using your results in parts (a)–(d)? Write your conjectures as conditional statements written in if-then form.

Math Practice

Construct Arguments
In part (d), does it matter which side you choose when finding r? Explain your reasoning.

Using the Circumcenter of a Triangle

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Vocabulary

VOCAB

concurrent, *p. 300*
point of concurrency, *p. 300*
circumcenter, *p. 300*
incenter, *p. 303*

When three or more lines, rays, or segments intersect in the same point, they are called **concurrent** lines, rays, or segments. The point of intersection of the lines, rays, or segments is called the **point of concurrency**.

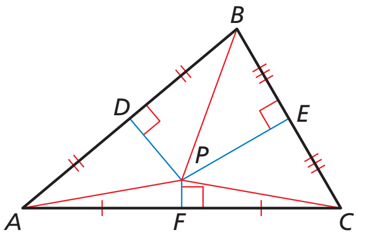
In a triangle, the three perpendicular bisectors are concurrent. The point of concurrency is the **circumcenter** of the triangle.

THEOREM

6.5 Circumcenter Theorem

The circumcenter of a triangle is equidistant from the vertices of the triangle.

If $\overline{PD}$, $\overline{PE}$, and $\overline{PF}$ are perpendicular bisectors, then $PA = PB = PC$.

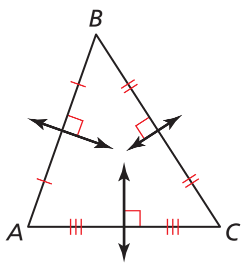


PROOF Circumcenter Theorem

Given $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$

Prove The perpendicular bisectors intersect in a point; that point is equidistant from A, B, and C.

Plan for Proof Show that P, the point of intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, also lies on the perpendicular bisector of $\overline{AC}$. Then show that point P is equidistant from the vertices of the triangle.

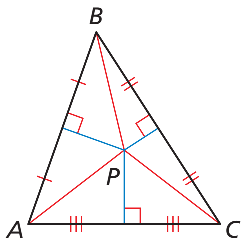


Plan in Action

STATEMENTS	REASONS
1. $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$	**1.** Given
2. The perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect at some point P.	**2.** Because the sides of a triangle cannot be parallel, these perpendicular bisectors must intersect in some point. Call it P.
3. Draw $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$.	**3.** Two Point Postulate
4. $PA = PB$, $PB = PC$	**4.** Perpendicular Bisector Theorem
5. $PA = PC$	**5.** Transitive Property of Equality
6. P is on the perpendicular bisector of $\overline{AC}$.	**6.** Converse of the Perpendicular Bisector Theorem
7. $PA = PB = PC$. So, P is equidistant from the vertices of the triangle.	**7.** From the results of Steps 4 and 5, and the definition of equidistant

STUDY TIP

Use diagrams like the one below to help visualize your proof.



EXAMPLE 1 Modeling Real Life

WATCH

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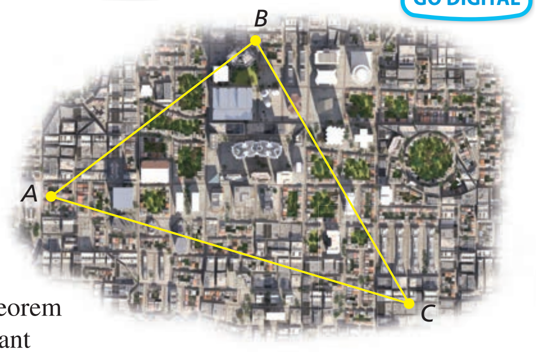
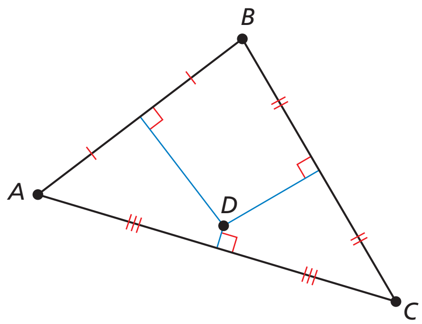
Three snack carts sell frozen yogurt at points A, B, and C in a city. Each of the three carts is the same distance from the frozen yogurt distributor. Find the location of the distributor.



SOLUTION

The distributor is equidistant from the three snack carts. The Circumcenter Theorem shows that you can find a point equidistant from three points by using the perpendicular bisectors of the triangle formed by the three points.

Copy points A, B, and C and connect them to draw $\triangle ABC$. Then use a ruler and protractor to draw the three perpendicular bisectors of $\triangle ABC$. The circumcenter D is the location of the distributor.



SELF-ASSESSMENT

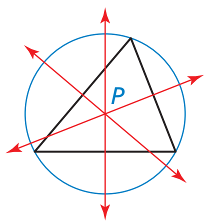
1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. Three snack carts sell hot pretzels at points A, B, and E. Each of the three carts is the same distance from the pretzel distributor. Find the location of the distributor.

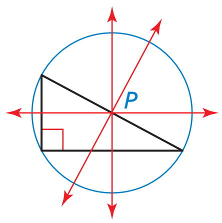
READING

The prefix *circum-* means "around" or "about," as in *circumference* (distance around a circle).

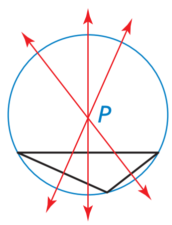
The circumcenter P is equidistant from the three vertices, so P is the center of a circle that passes through all three vertices. As shown below, the location of P depends on the type of triangle. The circle with center P is said to be *circumscribed* about the triangle.



Acute triangle
P is inside triangle.



Right triangle
P is on triangle.



Obtuse triangle
P is outside triangle.

CONSTRUCTION Circumscribing a Circle About a Triangle

WATCH

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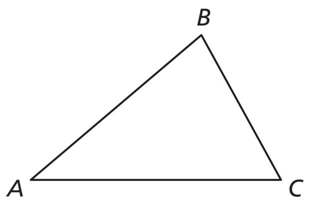
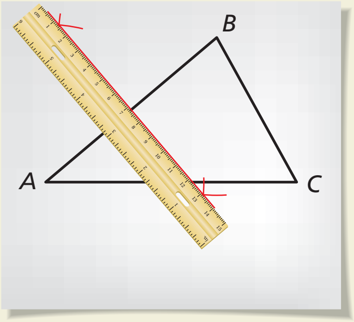
Use a compass and straightedge to construct a circle that is circumscribed about $\triangle ABC$.



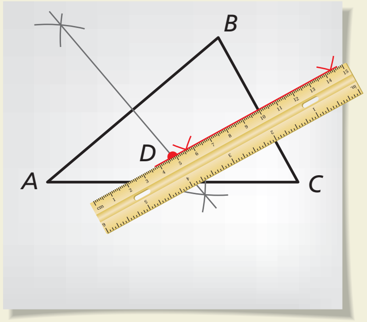
SOLUTION

Step 1



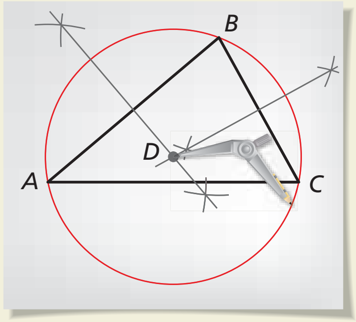
Draw a perpendicular bisector Draw the perpendicular bisector of $\overline{AB}$.

Step 2



Draw a perpendicular bisector Draw the perpendicular bisector of $\overline{BC}$. Label the intersection of the bisectors D. This is the circumcenter.

Step 3



Draw a circle Place the compass at D. Set the width by using any vertex of the triangle. This is the radius of the *circumcircle*. Draw the circle. It should pass through all three vertices A, B, and C.

EXAMPLE 2 Finding the Circumcenter of a Triangle

WATCH

Find the coordinates of the circumcenter of $\triangle ABC$ with vertices $A(0, 3)$, $B(0, -1)$, and $C(6, -1)$.

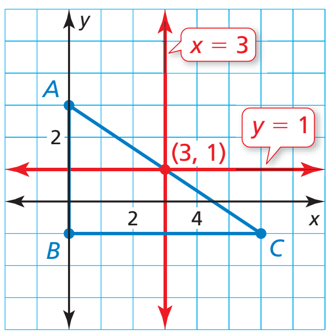
SOLUTION

Step 1 Graph $\triangle ABC$.

Step 2 Find equations of two perpendicular bisectors. Use the Slopes of Perpendicular Lines Theorem, which states that horizontal lines are perpendicular to vertical lines.

The midpoint of $\overline{AB}$ is $(0, 1)$. The line through $(0, 1)$ that is perpendicular to $\overline{AB}$ is $y = 1$.

The midpoint of $\overline{BC}$ is $(3, -1)$. The line through $(3, -1)$ that is perpendicular to $\overline{BC}$ is $x = 3$.



STUDY TIP

You need to find the equations of only *two* perpendicular bisectors. You can use the perpendicular bisector of the third side to verify your result.

Step 3 Find the point where $x = 3$ and $y = 1$ intersect. They intersect at $(3, 1)$.

▶ So, the coordinates of the circumcenter are $(3, 1)$.

Using the Incenter of a Triangle

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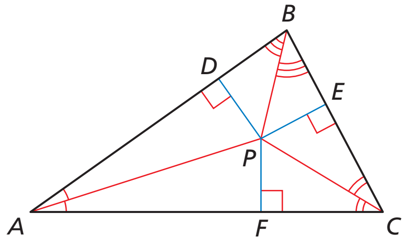
Just as a triangle has three perpendicular bisectors, it also has three angle bisectors. The angle bisectors of a triangle are also concurrent. This point of concurrency is the **incenter** of the triangle. For any triangle, the incenter always lies inside the triangle.

THEOREM

6.6 Incenter Theorem

The incenter of a triangle is equidistant from the sides of the triangle.

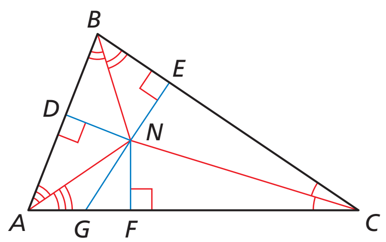
If $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ are angle bisectors of $\triangle ABC$, then $PD = PE = PF$.



Prove this Theorem Exercise 38, page 307

EXAMPLE 3 Using the Incenter of a Triangle

WATCH



In the figure shown, $ND = 5x - 1$ and $NE = 2x + 11$.

a. Find NF.

b. Can NG be equal to 18? Explain your reasoning.

SOLUTION

a. Point N is the incenter of $\triangle ABC$ because it is the point of concurrency of the three angle bisectors. So, by the Incenter Theorem, $ND = NE = NF$.

Step 1 Solve for x.

$ND = NE$	Incenter Theorem
$5x - 1 = 2x + 11$	Substitute.
$x = 4$	Solve for x.

Step 2 Find ND (or NE).

$$ND = 5x - 1 = 5(4) - 1 = 19$$

▶ So, because $ND = NF$, $NF = 19$.

b. Recall that the shortest distance between a point and a line is the length of a perpendicular segment. In this case, the perpendicular segment is $\overline{NF}$, which has a length of 19. Because $18 < 19$, NG cannot be equal to 18.

SELF-ASSESSMENT

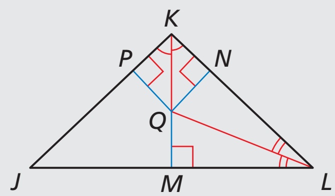
1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

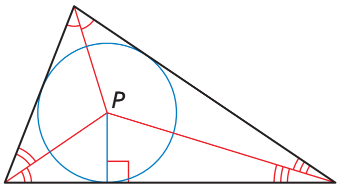
Find the coordinates of the circumcenter of the triangle with the given vertices.

2. $R(-2, 5)$, $S(-6, 5)$, $T(-2, -1)$

3. $W(-1, 4)$, $X(1, 4)$, $Y(1, -6)$

4. In the figure shown, $QM = 3x + 8$ and $QN = 7x + 2$. Find QP.





Because the incenter P is equidistant from the three sides of the triangle, a circle drawn using P as the center and the distance to one side of the triangle as the radius will just touch the other two sides of the triangle. The circle is said to be *inscribed* within the triangle.

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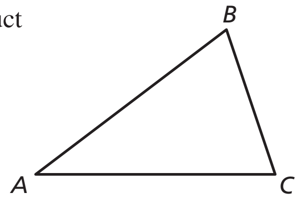
CONSTRUCTION Inscribing a Circle Within a Triangle

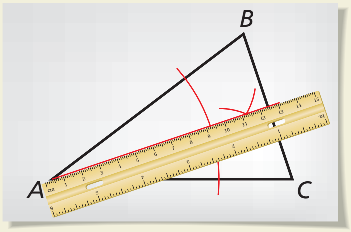
WATCH

Use a compass and straightedge to construct a circle that is inscribed within $\triangle ABC$.



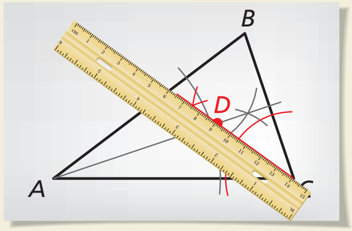
SOLUTION

Step 1



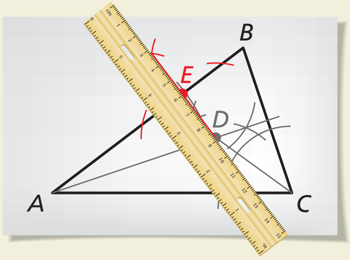
Draw an angle bisector Draw the angle bisector of $\angle A$.

Step 2



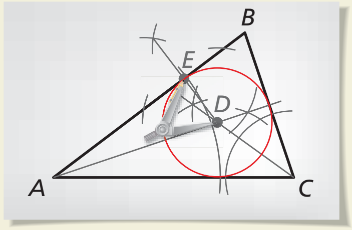
Draw an angle bisector Draw the angle bisector of $\angle C$. Label the intersection of the bisectors D. This is the incenter.

Step 3



Draw a perpendicular line Draw the perpendicular line from D to $\overline{AB}$. Label the point where it intersects $\overline{AB}$ as E.

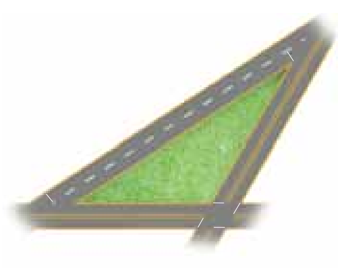
Step 4



Draw a circle Place the compass at D. Set the width to E. This is the radius of the *incircle*. Draw the circle. It should touch each side of the triangle.

EXAMPLE 4 Modeling Real Life

WATCH



City officials want to place a lamppost near the streets shown so that the lamppost is the same distance from all three streets. Should the lamppost be at the *circumcenter* or *incenter* of the triangular piece of land? Explain.

SOLUTION

Because the shape of the land is an obtuse triangle, the circumcenter lies outside the triangle and is not equidistant from the sides of the triangle. By the Incenter Theorem, the incenter of the triangle is equidistant from the sides of the triangle.

▶ So, the lamppost should be at the incenter of the triangular piece of land.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

5. Draw a sketch to show the location L of the lamppost in Example 4.

6.2 Practice WITH CalcChat® AND CalcView®

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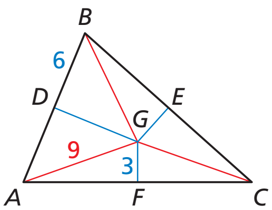
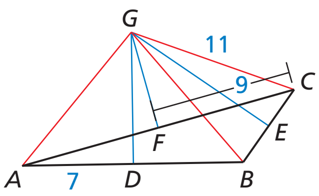
In Exercises 1 and 2, the perpendicular bisectors of $\triangle ABC$ intersect at point G and are shown in blue. Find the indicated measure.

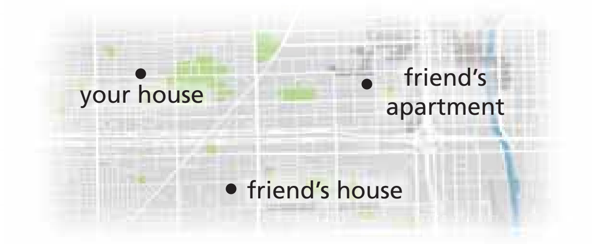
1. BG



2. GA



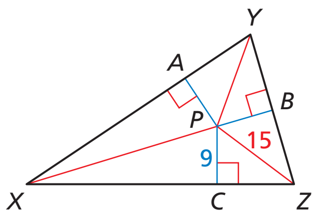
3. **MODELING REAL LIFE** You and two friends plan to walk your dogs together. You want the meeting place to be the same distance from each person's residence. Explain how you can use the diagram to locate the meeting place. *Example 1*



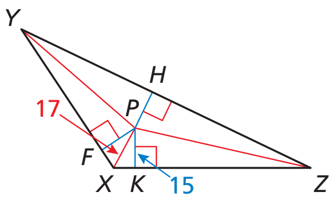
4. **MODELING REAL LIFE** You open a donation center the same distance from each of the three stores shown. Use the diagram to determine the location of the donation center.

In Exercises 5 and 6, the angle bisectors of $\triangle XYZ$ intersect at point P and are shown in red. Find the indicated measure.

5. PB



6. HP



In Exercises 7–10, find the coordinates of the circumcenter of the triangle with the given vertices. *Example 2*

7. $A(0, 0)$, $B(0, 8)$, $C(6, 0)$

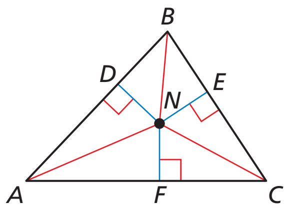
8. $A(2, 2)$, $B(2, 4)$, $C(8, 4)$

9. $H(-10, 7)$, $J(-6, 3)$, $K(-2, 3)$

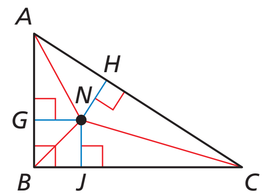
10. $L(3, -6)$, $M(5, -3)$, $N(8, -6)$

In Exercises 11–14, point N is the incenter of $\triangle ABC$. Use the given information to find the indicated measure. *Example 3*

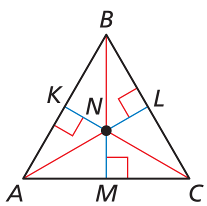
11. $ND = 6x - 2$
$NE = 3x + 7$
Find NF.



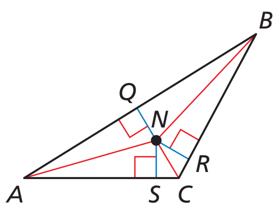
12. $NG = x + 3$
$NH = 2x - 3$
Find NJ.



13. $NK = 2x - 2$
$NL = -x + 10$
Find NM.



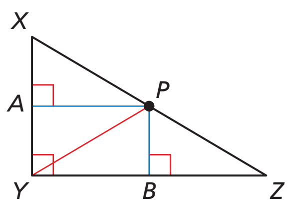
14. $NQ = 2x$
$NR = 3x - 2$
Find NS.



15. Point P is the circumcenter of $\triangle XYZ$. Use the given information to find PZ.

$PX = 3x + 2$

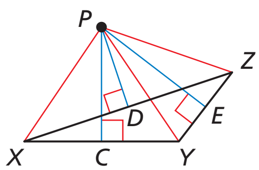
$PY = 4x - 8$



16. Point P is the circumcenter of $\triangle XYZ$. Use the given information to find PY.

$PX = 4x + 3$

$PZ = 6x - 11$



CONSTRUCTION **In Exercises 17–20, draw a triangle of the given type. Find the circumcenter. Then construct the circumscribed circle.**

17. right

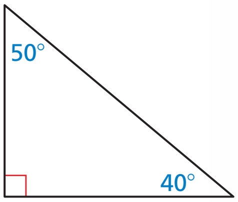
18. obtuse

19. acute isosceles

20. equilateral

CONSTRUCTION **In Exercises 21–24, copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.**

21.



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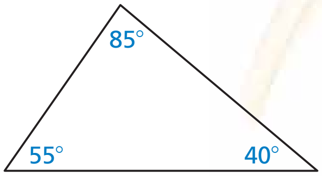

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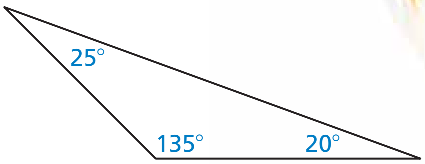

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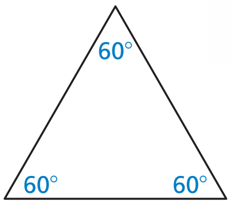
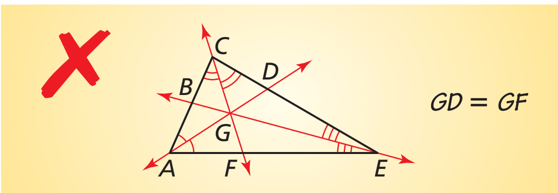
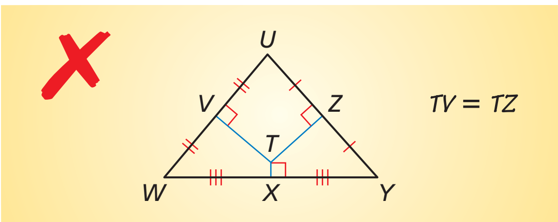


ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in identifying equal distances.**

25.



26.



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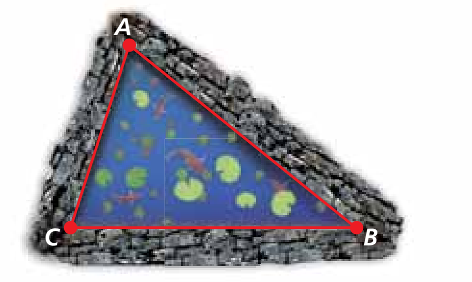
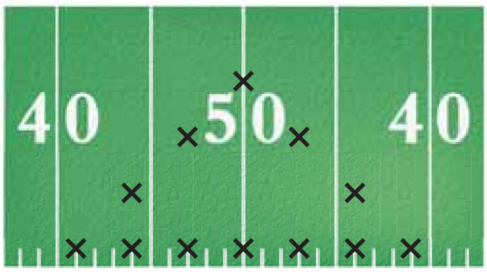
27. **MODELING REAL LIFE** You are placing a fountain in a triangular koi pond. You want the fountain to be the same distance from each side of the pond. Should the fountain be at the *circumcenter* or *incenter* of the triangular pond? Explain.



28. **MODELING REAL LIFE** A marching band director wants a soloist to be the same distance from each side of the triangular formation shown below. Should the soloist be at the *cirumcenter* or *incenter* of the triangular formation? Explain.



CRITICAL THINKING **In Exercises 29–32, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.**

29. The circumcenter of a scalene triangle is __________ inside the triangle.

30. If the perpendicular bisector of one side of a triangle intersects the opposite vertex, then the triangle is __________ isosceles.

31. The perpendicular bisectors of a triangle intersect at a point that is _________ equidistant from the midpoints of the sides of the triangle.

32. The angle bisectors of a triangle intersect at a point that is _________ equidistant from the sides of the triangle.

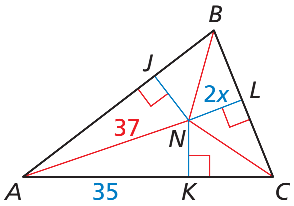
CRITICAL THINKING **In Exercises 33 and 34, find the coordinates of the circumcenter of the triangle with the given vertices.**

33. $A(2, 5)$, $B(6, 6)$, $C(12, 3)$

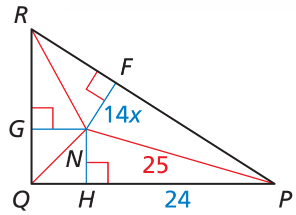
34. $D(-9, -5)$, $E(-5, -9)$, $F(-2, -2)$

MP STRUCTURE In Exercises 35 and 36, find the value of x that makes point N the incenter of the triangle.

35.



36.

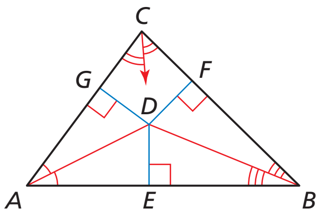


37. **PROOF** Where is the circumcenter located in any right triangle? Write a coordinate proof of this result.

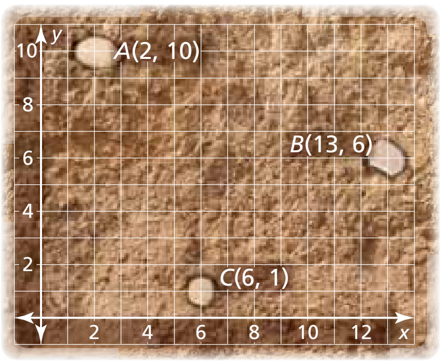
38. **PROVING A THEOREM** Write a proof of the Incenter Theorem.

Given $\triangle ABC$, $\overline{AD}$ bisects $\angle CAB$, $\overline{BD}$ bisects $\angle CBA$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, and $\overline{DG} \perp \overline{CA}$

Prove The angle bisectors intersect at D, which is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.



39. **MP PROBLEM SOLVING** Archaeologists find three stones. They believe that the stones were once part of a circle of stones with a community fire pit at its center. They mark the locations of stones A, B, and C on a coordinate plane, where distances are measured in feet.



a. Explain how archaeologists can use a sketch to estimate the center of the circle of stones.

b. Copy the diagram and find the approximate coordinates of the point at which the archaeologists should look for the fire pit.

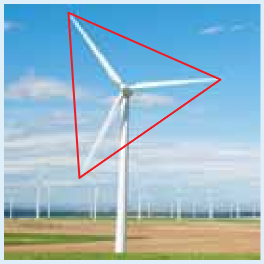
40. **PERFORMANCE TASK** You plan to install and furnish the largest circular patio possible inside a triangular yard with side lengths of 9 meters, 12 meters, and 15 meters.

a. Create a blueprint. Include the center and radius of the circle. Estimate the circumference and the area of the circle.

b. Research costs of materials and labor. Then prepare an itemized cost estimate for installing and furnishing the patio.

41. **MP REASONING** Is it possible for the incenter and the circumcenter of a triangle to be the same point? Use diagrams to support your reasoning.

42. **HOW DO YOU SEE IT?** The arms of the windmill are the angle bisectors of the red triangle. What point of concurrency is the point that connects the three arms?

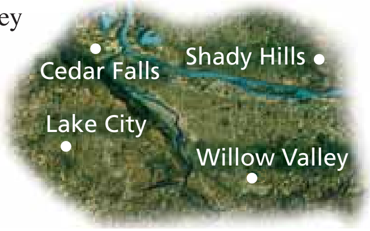


43. **CRITICAL THINKING** Explain why the incenter of a triangle is always located inside the triangle.

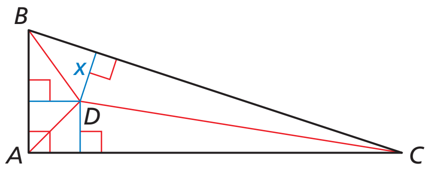
44. **MP REPEATED REASONING** Use reflections to show that the three lines of symmetry of an equilateral triangle are perpendicular bisectors of the sides of the triangle.

45. **MP USING TOOLS** Cut the largest circle possible from an isosceles triangle made of paper whose sides are 8 inches, 12 inches, and 12 inches. Find the radius of the circle. State whether you used perpendicular bisectors or angle bisectors.

46. **DIG DEEPER** A high school is being built to accommodate the towns shown. Each town agrees that the school should be an equal distance from each of the four towns. Is there a single point where they could agree to build the school? If so, find it. If not, explain why not. Justify your answer with a diagram.



47. **CRITICAL THINKING** Point D is the incenter of $\triangle ABC$. Write an expression for the length x in terms of the three side lengths AB, AC, and BC.



48. **THOUGHT PROVOKING**
You are asked to draw a triangle and its perpendicular bisectors and angle bisectors.

a. For which type of triangle would you need the fewest segments? What is the minimum number of segments you would need? Explain.

b. For which type of triangle would you need the most segments? What is the maximum number of segments you would need? Explain.

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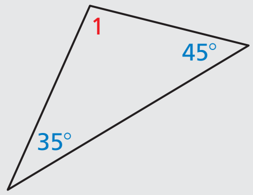
49. Determine whether $\triangle QRS$ and $\triangle TUV$ with the given vertices are congruent. Use transformations to explain your reasoning.

$Q(1, 2)$, $R(3, 5)$, $S(5, 1)$

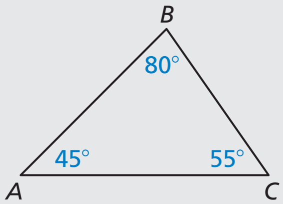
$T(1, 1)$, $U(3, 3)$, $V(5, 0)$

50. **MODELING REAL LIFE** The largest concentrated solar power plant in the world is the Noor Complex, located in the Sahara Desert. It cost 3.9 billion dollars to construct. Use this information to write two true conditional statements.

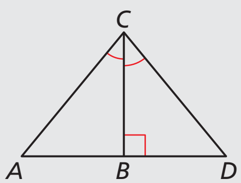
51. Find $m\angle 1$. Then classify the triangle by its angles.



52. Copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.

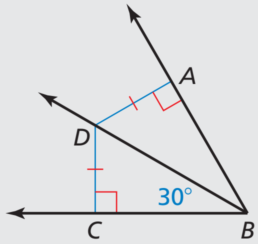


53. Explain how to prove that $\angle A \cong \angle D$.

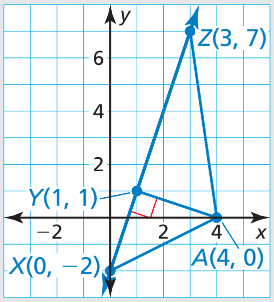


54. Factor $5n^5 + 25n^3$.

55. Find $m\angle ABD$.



56. Find the distance from point A to $\overleftrightarrow{XZ}$.



57. A triangle has vertices $A(0, 0)$, $B(8, 12)$, and $C(16, 0)$. Prove that $\triangle ABC$ is isosceles.

58. The endpoints of $\overline{AB}$ are $A(-3, 5)$ and $B(3, 5)$. Find the coordinates of the midpoint M. Then find AB.

59. For $h(x) = -7x$, find the value of x for which $h(x) = 42$.

60. Determine whether the table represents a *linear* or *nonlinear* function. Explain.

x	5	7	9	11
y	−4	−1	2	5