6.2 Bisectors of Triangles

Learning Target
Use bisectors of triangles.

Success Criteria
• I can find the circumcenter and incenter of a triangle.
• I can circumscribe a circle about a triangle.
• I can inscribe a circle within a triangle.
• I can use points of concurrency to solve real-life problems.

EXPLORE IT! Analyzing Bisectors of Triangles

Work with a partner.

a. Use technology to draw any triangle and the perpendicular bisectors of all three sides of the triangle. What do you notice? What happens when you move the vertices of the triangle?

b. Draw the circle with its center at the intersection of the perpendicular bisectors that passes through a vertex of the triangle. What do you notice? What does this mean?

c. Use technology to draw a different triangle and the angle bisectors of all three angles of the triangle. What do you notice? What happens when you move the vertices of the triangle?

d. Find the distance \( r \) between the intersection of the angle bisectors and one of the sides of the triangle. Draw the circle with its center at the intersection of the angle bisectors and radius \( r \). What do you notice? What does this mean?

e. What conjectures can you make using your results in parts (a)–(d)? Write your conjectures as conditional statements written in if-then form.
Using the Circumcenter of a Triangle

When three or more lines, rays, or segments intersect in the same point, they are called **concurrent** lines, rays, or segments. The point of intersection of the lines, rays, or segments is called the **point of concurrency**.

In a triangle, the three perpendicular bisectors are concurrent. The point of concurrency is the **circumcenter** of the triangle.

**THEOREM**

6.5 **Circumcenter Theorem**

The circumcenter of a triangle is equidistant from the vertices of the triangle.

If $PD, PE,$ and $PF$ are perpendicular bisectors, then $PA = PB = PC$.

**PROOF** Circumcenter Theorem

**Given** $\triangle ABC$; the perpendicular bisectors of $AB, BC,$ and $AC$

**Prove** The perpendicular bisectors intersect in a point; that point is equidistant from $A, B,$ and $C$.

**Plan for Proof**

Show that $P$, the point of intersection of the perpendicular bisectors of $AB$ and $BC$, also lies on the perpendicular bisector of $AC$. Then show that point $P$ is equidistant from the vertices of the triangle.

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\triangle ABC$, the perpendicular bisectors of $AB, BC,$ and $AC$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. The perpendicular bisectors of $AB$ and $BC$ intersect at some point $P$.</td>
<td>2. Because the sides of a triangle cannot be parallel, these perpendicular bisectors must intersect in some point. Call it $P$.</td>
</tr>
<tr>
<td>3. Draw $PA, PB,$ and $PC$.</td>
<td>3. Two Point Postulate</td>
</tr>
<tr>
<td>4. $PA = PB, PB = PC$</td>
<td>4. Perpendicular Bisector Theorem</td>
</tr>
<tr>
<td>5. $PA = PC$</td>
<td>5. Transitive Property of Equality</td>
</tr>
<tr>
<td>6. $P$ is on the perpendicular bisector of $AC$.</td>
<td>6. Converse of the Perpendicular Bisector Theorem</td>
</tr>
<tr>
<td>7. $PA = PB = PC$. So, $P$ is equidistant from the vertices of the triangle.</td>
<td>7. From the results of Steps 4 and 5, and the definition of equidistant</td>
</tr>
</tbody>
</table>
EXAMPLE 1  Modeling Real Life

Three snack carts sell frozen yogurt at points $A$, $B$, and $C$ in a city. Each of the three carts is the same distance from the frozen yogurt distributor. Find the location of the distributor.

SOLUTION

The distributor is equidistant from the three snack carts. The Circumcenter Theorem shows that you can find a point equidistant from three points by using the perpendicular bisectors of the triangle formed by the three points.

Copy points $A$, $B$, and $C$ and connect them to draw $\triangle ABC$. Then use a ruler and protractor to draw the three perpendicular bisectors of $\triangle ABC$. The circumcenter $D$ is the location of the distributor.

The circumcenter $P$ is equidistant from the three vertices, so $P$ is the center of a circle that passes through all three vertices. As shown below, the location of $P$ depends on the type of triangle. The circle with center $P$ is said to be circumscribed about the triangle.

READING

The prefix *circum-* means “around” or “about,” as in circumference (distance around a circle).
CONSTRUCTION  Circumscribing a Circle About a Triangle

Use a compass and straightedge to construct a circle that is circumscribed about $\triangle ABC$.

SOLUTION

Step 1

Draw a perpendicular bisector Draw the perpendicular bisector of $\overline{AB}$.

Step 2

Draw a perpendicular bisector Draw the perpendicular bisector of $\overline{BC}$. Label the intersection of the bisectors $D$. This is the circumcenter.

Step 3

Draw a circle Place the compass at $D$. Set the width by using any vertex of the triangle. This is the radius of the circumcircle. Draw the circle. It should pass through all three vertices $A$, $B$, and $C$.

EXAMPLE 2  Finding the Circumcenter of a Triangle

Find the coordinates of the circumcenter of $\triangle ABC$ with vertices $A(0, 3)$, $B(0, -1)$, and $C(6, -1)$.

SOLUTION

Step 1  Graph $\triangle ABC$.

Step 2  Find equations of two perpendicular bisectors. Use the Slopes of Perpendicular Lines Theorem, which states that horizontal lines are perpendicular to vertical lines.

The midpoint of $\overline{AB}$ is $(0, 1)$. The line through $(0, 1)$ that is perpendicular to $\overline{AB}$ is $y = 1$.

The midpoint of $\overline{BC}$ is $(3, -1)$. The line through $(3, -1)$ that is perpendicular to $\overline{BC}$ is $x = 3$.

Step 3  Find the point where $x = 3$ and $y = 1$ intersect. They intersect at $(3, 1)$.

So, the coordinates of the circumcenter are $(3, 1)$.

302  Chapter 6  Relationships Within Triangles
Using the Incenter of a Triangle

Just as a triangle has three perpendicular bisectors, it also has three angle bisectors. The angle bisectors of a triangle are also concurrent. This point of concurrency is the incenter of the triangle. For any triangle, the incenter always lies inside the triangle.

**THEOREM**

**6.6 Incenter Theorem**

The incenter of a triangle is equidistant from the sides of the triangle.

If \( AP, BP, \) and \( CP \) are angle bisectors of \( \triangle ABC \), then \( PD = PE = PF \).

*Prove this Theorem* Exercise 38, page 307

**EXAMPLE 3 Using the Incenter of a Triangle**

In the figure shown, \( ND = 5x - 1 \) and \( NE = 2x + 11 \).

a. Find \( NF \).

b. Can \( NG \) be equal to 18? Explain your reasoning.

**SOLUTION**

a. Point \( N \) is the incenter of \( \triangle ABC \) because it is the point of concurrency of the three angle bisectors. So, by the Incenter Theorem, \( ND = NE = NF \).

**Step 1** Solve for \( x \).

\[
ND = NE \quad \text{Incenter Theorem}
\]
\[
5x - 1 = 2x + 11 \quad \text{Substitute.}
\]
\[
x = 4 \quad \text{Solve for } x.
\]

**Step 2** Find \( ND \) (or \( NE \)).

\[
ND = 5x - 1 = 5(4) - 1 = 19
\]

So, because \( ND = NF, NF = 19 \).

b. Recall that the shortest distance between a point and a line is the length of a perpendicular segment. In this case, the perpendicular segment is \( NF \), which has a length of 19. Because 18 < 19, \( NG \) cannot be equal to 18.

**SELF-ASSESSMENT**

1. I do not understand.
2. I can do it with help.
3. I can do it on my own.
4. I can teach someone else.

Find the coordinates of the circumcenter of the triangle with the given vertices.

2. \( R(-2, 5), S(-6, 5), T(-2, -1) \)

3. \( W(-1, 4), X(1, 4), Y(1, -6) \)

4. In the figure shown, \( QM = 3x + 8 \) and \( QN = 7x + 2 \). Find \( QP \).
Because the incenter $P$ is equidistant from the three sides of the triangle, a circle drawn using $P$ as the center and the distance to one side of the triangle as the radius will just touch the other two sides of the triangle. The circle is said to be inscribed within the triangle.

**CONSTRUCTION**  **Inscribing a Circle Within a Triangle**

Use a compass and straightedge to construct a circle that is inscribed within $\triangle ABC$.

**SOLUTION**

**Step 1**

**Draw an angle bisector** Draw the angle bisector of $\angle A$.

**Step 2**

**Draw an angle bisector** Draw the angle bisector of $\angle C$. Label the intersection of the bisectors $D$. This is the incenter.

**Step 3**

**Draw a perpendicular line** Draw the perpendicular line from $D$ to $AB$. Label the point where it intersects $AB$ as $E$.

**Step 4**

**Draw a circle** Place the compass at $D$. Set the width to $E$. This is the radius of the incircle. Draw the circle. It should touch each side of the triangle.

**EXAMPLE 4**  **Modeling Real Life**

City officials want to place a lamppost near the streets shown so that the lamppost is the same distance from all three streets. Should the lamppost be at the circumcenter or incenter of the triangular piece of land? Explain.

**SOLUTION**

Because the shape of the land is an obtuse triangle, the circumcenter lies outside the triangle and is not equidistant from the sides of the triangle. By the Incenter Theorem, the incenter of the triangle is equidistant from the sides of the triangle.

So, the lamppost should be at the incenter of the triangular piece of land.

**SELF-ASSESSMENT**

1. I do not understand.
2. I can do it with help.
3. I can do it on my own.
4. I can teach someone else.

5. Draw a sketch to show the location $L$ of the lamppost in Example 4.
In Exercises 1 and 2, the perpendicular bisectors of \( \triangle ABC \) intersect at point \( G \) and are shown in blue. Find the indicated measure.

1. \( BG \)

2. \( GA \)

**3. MODELING REAL LIFE** You and two friends plan to walk your dogs together. You want the meeting place to be the same distance from each person’s residence. Explain how you can use the diagram to locate the meeting place.  

**Example 1**

**4. MODELING REAL LIFE** You open a donation center the same distance from each of the three stores shown. Use the diagram to determine the location of the donation center.

In Exercises 5 and 6, the angle bisectors of \( \triangle XYZ \) intersect at point \( P \) and are shown in red. Find the indicated measure.

5. \( PB \)

6. \( HP \)

In Exercises 7–10, find the coordinates of the circumcenter of the triangle with the given vertices.

**Example 2**

7. \( A(0, 0), B(0, 8), C(6, 0) \)

8. \( A(2, 2), B(2, 4), C(8, 4) \)

9. \( H(-10, 7), J(-6, 3), K(-2, 3) \)

10. \( L(3, -6), M(5, -3), N(8, -6) \)

In Exercises 11–14, point \( N \) is the incenter of \( \triangle ABC \). Use the given information to find the indicated measure.

**Example 3**

11. \( ND = 6x - 2 \)

12. \( NG = x + 3 \)

13. \( NK = 2x - 2 \)

14. \( NQ = 2x \)

Find \( NF \). Find \( NJ \).

15. Point \( P \) is the circumcenter of \( \triangle XYZ \). Use the given information to find \( PZ \).

\( PX = 3x + 2 \)

\( PY = 4x - 8 \)

16. Point \( P \) is the circumcenter of \( \triangle XYZ \). Use the given information to find \( PY \).

\( PX = 4x + 3 \)

\( PZ = 6x - 11 \)
CONSTRUCTION In Exercises 17–20, draw a triangle of the given type. Find the circumcenter. Then construct the circumscribed circle.

17. right  
18. obtuse  
19. acute isosceles  
20. equilateral

CONSTRUCTION In Exercises 21–24, copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.

21. 

22. 

23. 

24. 

ERROR ANALYSIS In Exercises 25 and 26, describe and correct the error in identifying equal distances.

25. 

26. 

CRITICAL THINKING In Exercises 29–32, complete the statement with always, sometimes, or never. Explain your reasoning.

29. The circumcenter of a scalene triangle is ________ inside the triangle.

30. If the perpendicular bisector of one side of a triangle intersects the opposite vertex, then the triangle is ________ isosceles.

31. The perpendicular bisectors of a triangle intersect at a point that is ________ equidistant from the midpoints of the sides of the triangle.

32. The angle bisectors of a triangle intersect at a point that is ________ equidistant from the sides of the triangle.

CRITICAL THINKING In Exercises 33 and 34, find the coordinates of the circumcenter of the triangle with the given vertices.

33. A(2, 5), B(6, 6), C(12, 3)

34. D(−9, −5), E(−5, −9), F(−2, −2)

27. MODELING REAL LIFE You are placing a fountain in a triangular koi pond. You want the fountain to be the same distance from each side of the pond. Should the fountain be at the circumcenter or incenter of the triangular pond? Explain.

28. MODELING REAL LIFE A marching band director wants a soloist to be the same distance from each side of the triangular formation shown below. Should the soloist be at the circumcenter or incenter of the triangular formation? Explain.
MP STRUCTURE In Exercises 35 and 36, find the value of \( x \) that makes point \( N \) the incenter of the triangle.

35. \[ \triangle ABC, \overline{AD} \text{ bisects } \angle CAB, \]
   \[ \overline{BD} \text{ bisects } \angle CBA, \]
   \[ \overline{DE} \perp \overline{AB}, \overline{DF} \perp \overline{BC}, \text{ and } \overline{DG} \perp \overline{CA} \]
   
36. \[ \triangle JKL, \overline{RL} \text{ bisects } \angle JKL, \]
   \[ \overline{JN} \perp \overline{KL}, \overline{JF} \perp \overline{LB}, \text{ and } \overline{GJ} \perp \overline{KC} \]

37. PROOF Where is the circumcenter located in any right triangle? Write a coordinate proof of this result.

38. PROVING A THEOREM Write a proof of the Incenter Theorem.
   Given \( \triangle ABC, \overline{AD} \text{ bisects } \angle CAB, \overline{BD} \text{ bisects } \angle CBA, \overline{DE} \perp \overline{AB}, \overline{DF} \perp \overline{BC}, \text{ and } \overline{DG} \perp \overline{CA} \)
   
   Prove The angle bisectors intersect at \( D \), which is equidistant from \( \overline{AB}, \overline{BC}, \) and \( \overline{CA} \).

39. MP PROBLEM SOLVING Archaeologists find three stones. They believe that the stones were once part of a circle of stones with a community fire pit at its center. They mark the locations of stones \( A, B, \) and \( C \) on a coordinate plane, where distances are measured in feet.

   a. Explain how archaeologists can use a sketch to estimate the center of the circle of stones.
   b. Copy the diagram and find the approximate coordinates of the point at which the archaeologists should look for the fire pit.

40. PERFORMANCE TASK You plan to install and furnish the largest circular patio possible inside a triangular yard with side lengths of 9 meters, 12 meters, and 15 meters.

   a. Create a blueprint. Include the center and radius of the circle. Estimate the circumference and the area of the circle.
   b. Research costs of materials and labor. Then prepare an itemized cost estimate for installing and furnishing the patio.

41. MP REASONING Is it possible for the incenter and the circumcenter of a triangle to be the same point? Use diagrams to support your reasoning.

42. HOW DO YOU SEE IT? The arms of the windmill are the angle bisectors of the red triangle. What point of concurrency is the point that connects the three arms?

43. CRITICAL THINKING Explain why the incenter of a triangle is always located inside the triangle.

44. MP REPEATED REASONING Use reflections to show that the three lines of symmetry of an equilateral triangle are perpendicular bisectors of the sides of the triangle.

45. MP USING TOOLS Cut the largest circle possible from an isosceles triangle made of paper whose sides are 8 inches, 12 inches, and 12 inches. Find the radius of the circle. State whether you used perpendicular bisectors or angle bisectors.

46. DIG DEEPER A high school is being built to accommodate the towns shown. Each town agrees that the school should be an equal distance from each of the four towns. Is there a single point where they could agree to build the school? If so, find it. If not, explain why not. Justify your answer with a diagram.
47. **CRITICAL THINKING** Point $D$ is the incenter of $\triangle ABC$. Write an expression for the length $x$ in terms of the three side lengths $AB$, $AC$, and $BC$.

![Diagram of incenter](image)

48. **THOUGHT PROVOKING** You are asked to draw a triangle and its perpendicular bisectors and angle bisectors.

   a. For which type of triangle would you need the fewest segments? What is the minimum number of segments you would need? Explain.

   b. For which type of triangle would you need the most segments? What is the maximum number of segments you would need? Explain.

### REVIEW & REFRESH

49. Determine whether $\triangle QRS$ and $\triangle TUV$ with the given vertices are congruent. Use transformations to explain your reasoning.

   - $Q(1, 2), R(3, 5), S(5, 1)$
   - $T(1, 1), U(3, 3), V(5, 0)$

50. **MODELING REAL LIFE** The largest concentrated solar power plant in the world is the Noor Complex, located in the Sahara Desert. It cost 3.9 billion dollars to construct. Use this information to write two true conditional statements.

51. Find $m \angle 1$. Then classify the triangle by its angles.

![Triangle with angles](image)

52. Copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.

![Triangle with angles](image)

53. Explain how to prove that $\angle A \equiv \angle D$.

![Triangle with angles](image)

54. Factor $5n^3 + 25n^2$.

55. Find $m \angle ABD$.

56. Find the distance from point $A$ to $\overline{XZ}$.

![Distance calculation](image)

57. A triangle has vertices $A(0, 0)$, $B(8, 12)$, and $C(16, 0)$. Prove that $\triangle ABC$ is isosceles.

58. The endpoints of $\overline{AB}$ are $A(−3, 5)$ and $B(3, 5)$. Find the coordinates of the midpoint $M$. Then find $AB$.

59. For $h(x) = −7x$, find the value of $x$ for which $h(x) = 42$.

60. Determine whether the table represents a linear or nonlinear function. Explain.

<table>
<thead>
<tr>
<th>$x$</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>−4</td>
<td>−1</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>