2.3 Arithmetic Sequences

GO DIGITAL

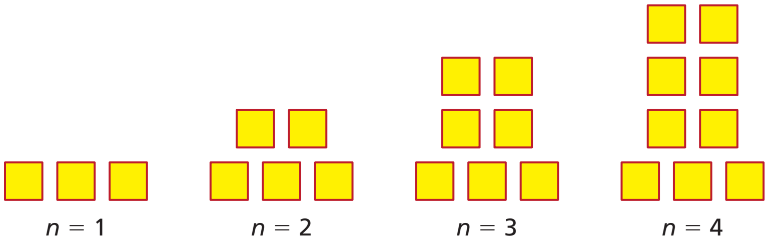
Learning Target Understand the concept of arithmetic sequences.

Success Criteria

- I can write the terms of arithmetic sequences.
- I can graph arithmetic sequences.
- I can identify arithmetic sequences.
- I can translate between recursive and explicit rules.

EXPLORE IT! Describing Patterns Involving Squares

Work with a partner. Use the figures below.



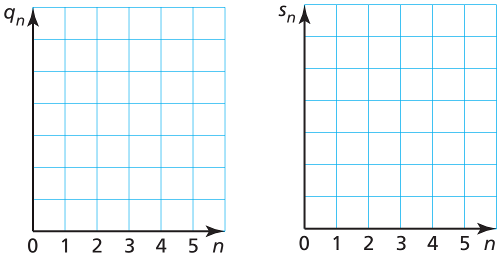
Math Practice

Analyze Givens
Be sure you can identify the important quantities in a problem. What do you think n represents in this situation?

a. Complete the table. What do you notice?

n	1	2	3	4
Number of squares, q_n	3			
Number of sides, s_n	12			

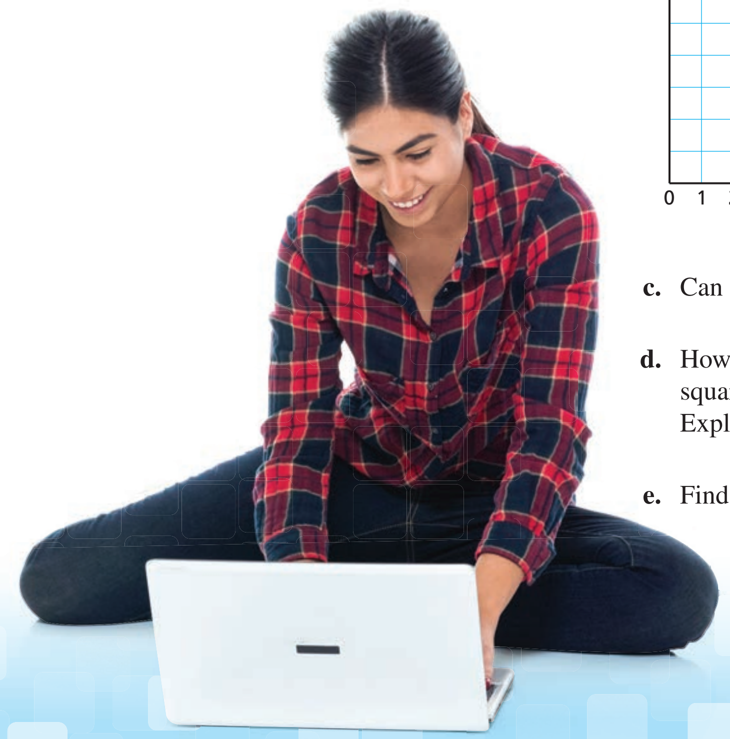
b. Graph the data in the table. What do you notice?



c. Can you write an equation that represents each graph? Explain.

d. How can you extend the patterns to find the number of squares or the number of sides for greater values of n? Explain your reasoning.

e. Find the number of squares and the number of sides when $n = 20$.



Vocabulary AZ VOCAB

sequence, *p. 90*
term, *p. 90*
arithmetic sequence, *p. 90*
common difference, *p. 90*
explicit rule, *p. 93*
recursive rule, *p. 93*

Writing the Terms of Arithmetic Sequences

A **sequence** is an ordered list of numbers. Each number in a sequence is called a **term**. Each term a_n has a specific position n in the sequence.

$$5,\quad 10,\quad 15,\quad 20,\quad 25, \ldots, a_n, \ldots$$

1st position

3rd position

*n*th position

KEY IDEA

Arithmetic Sequence

In an **arithmetic sequence**, the difference between each pair of consecutive terms is the same. This difference is called the **common difference**. Each term is found by adding the common difference to the previous term.

$$5,\quad 10,\quad 15,\quad 20, \ldots$$

Terms of an arithmetic sequence

$+5 \quad +5 \quad +5$ ← common difference

READING

An ellipsis (. . .) is a series of dots that indicates an intentional omission of information. In mathematics, the . . . notation means "and so forth." The ellipsis indicates that there are more terms in the sequence that are not shown.

EXAMPLE 1 Extending an Arithmetic Sequence

WATCH

Write the next three terms of the arithmetic sequence.

$$-7, -14, -21, -28, \ldots$$

SOLUTION

Use a table to organize the terms and find the pattern.

Position	1	2	3	4
Term	−7	−14	−21	−28

$+(-7) \quad +(-7) \quad +(-7)$ ←

Each term is 7 less than the previous term. So, the common difference is −7.

Add −7 to a term to find the next term.

Position	1	2	3	4	5	6	7
Term	−7	−14	−21	−28	−35	−42	−49

$+(-7) \quad +(-7) \quad +(-7)$

▶ The next three terms are −35, −42, and −49.

SELF-ASSESSMENT

1 I do not understand.
2 I can do it with help.
3 I can do it on my own.
4 I can teach someone else.

1. **WRITING** Explain how to find the common difference of an arithmetic sequence.
2. **OPEN-ENDED** Give an example of an arithmetic sequence. Then give an example of a sequence that is not arithmetic but has the same first term as the arithmetic sequence you wrote.

Write the next three terms of the arithmetic sequence.

3. $-12, 0, 12, 24, \ldots$
4. $0.2, 0.6, 1, 1.4, \ldots$
5. $4, 3\frac{3}{4}, 3\frac{1}{2}, 3\frac{1}{4}, \ldots$

Graphing Arithmetic Sequences

To graph a sequence, let a term's position number n in the sequence be the x-value. The term a_n is the corresponding y-value. Plot the ordered pairs (n, a_n).

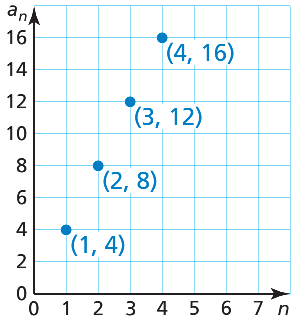
EXAMPLE 2 Graphing an Arithmetic Sequence

Graph the arithmetic sequence 4, 8, 12, 16, ... What do you notice?

SOLUTION

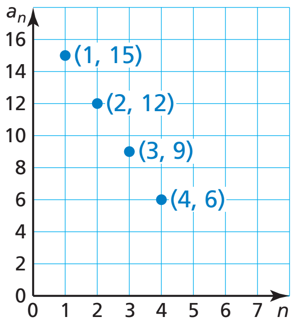
Make a table. Then plot the ordered pairs (n, a_n).

Position, n	**Term, a_n**
1	4
2	8
3	12
4	16



▶ The points lie on a line.

EXAMPLE 3 Identifying an Arithmetic Sequence from a Graph



Does the graph represent an arithmetic sequence? Explain.

SOLUTION

Make a table to organize the ordered pairs. Then determine whether there is a common difference.

Position, n	1	2	3	4
Term, a_n	15	12	9	6

$+(-3)$ $+(-3)$ $+(-3)$

Each term is 3 less than the previous term. So, the common difference is -3.

▶ Consecutive terms have a common difference of -3. So, the graph represents the arithmetic sequence 15, 12, 9, 6, ...

SELF-ASSESSMENT

1 I do not understand. **2** I can do it with help. **3** I can do it on my own. **4** I can teach someone else.

Graph the arithmetic sequence. What do you notice?

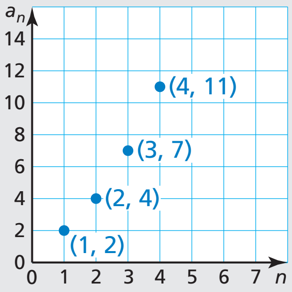
6. 3, 12, 21, 30, . . .

7. 4, 2, 0, -2, . . .

8. 1, 0.8, 0.6, 0.4, . . .

Determine whether the graph represents an arithmetic sequence. Explain.

9.



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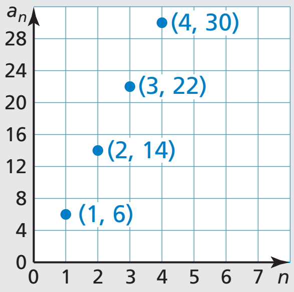

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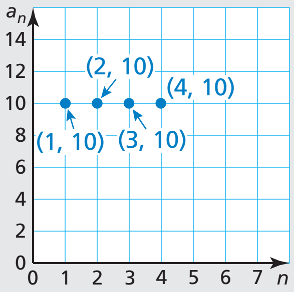


Writing Arithmetic Sequences as Functions

Because consecutive terms of an arithmetic sequence have a common difference, the sequence has a constant rate of change. So, the points represented by any arithmetic sequence lie on a line. You can use the first term and the common difference to write a linear function that describes an arithmetic sequence. For example, let $a_1 = 4$ and $d = 3$.

Position, n	Term, a_n	Written using a_1 and d	Numbers
1	first term, a_1	a_1	4
2	second term, a_2	$a_1 + d$	$4 + 3 = 7$
3	third term, a_3	$a_1 + 2d$	$4 + 2(3) = 10$
4	fourth term, a_4	$a_1 + 3d$	$4 + 3(3) = 13$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	nth term, a_n	$a_1 + (n - 1)d$	$4 + (n - 1)(3)$

ANOTHER WAY

An *arithmetic sequence* is a linear function whose domain is the set of positive integers. You can think of d as the slope and $(1, a_1)$ as a point on the graph of the function. An equation in point-slope form of the function is

$$a_n - a_1 = d(n - 1).$$

This equation can be rewritten as

$$a_n = a_1 + (n - 1)d$$

or

$$f(n) = a_1 + (n - 1)d.$$

KEY IDEA

Equation for an Arithmetic Sequence

Let a_n be the nth term of an arithmetic sequence with first term a_1 and common difference d. The nth term is given by $a_n = a_1 + (n - 1)d$.

EXAMPLE 4 Finding the *n*th Term of an Arithmetic Sequence

WATCH

Write an equation for the nth term of the arithmetic sequence 55, 40, 25, 10, ... Then find a_{16}.

SOLUTION

The first term is 55, and the common difference is −15.

$a_n = a_1 + (n - 1)d$ Equation for an arithmetic sequence

$a_n = 55 + (n - 1)(-15)$ Substitute 55 for a_1 and −15 for d.

$a_n = -15n + 70$ Simplify.

Use the equation to find the 16th term.

$a_n = -15n + 70$ Write the equation.

$a_{16} = -15(16) + 70$ Substitute 16 for n.

$= -170$ Simplify.

▶ The 16th term of the arithmetic sequence is −170.

STUDY TIP

Notice that the equation in Example 4 is of the form $y = mx + b$, where y is replaced by a_n and x is replaced by n.

SELF-ASSESSMENT

1 I do not understand. **2** I can do it with help. **3** I can do it on my own. **4** I can teach someone else.

Write an equation for the nth term of the arithmetic sequence. Then find a_{25}.

12. 8, 16, 24, 32, . . .

13. 1, 0, −1, −2, . . .

14. 4, $5\frac{1}{2}$, 7, $8\frac{1}{2}$, . . .

Writing Recursive Rules

So far, you have defined arithmetic sequences *explicitly*. An **explicit rule** gives a_n as a function of the term's position number n in the sequence. For example, an explicit rule for the arithmetic sequence 3, 5, 7, 9, . . . is $a_n = 3 + 2(n - 1)$, or $a_n = 2n + 1$.

Now, you will define arithmetic sequences *recursively*. A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

WORDS AND MATH

The prefix *re-* often indicates repetition.

KEY IDEA

Recursive Equation for an Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Math Practice

Make Sense of Quantities

When working with sequences, be sure you understand which term is represented by an expression such as a_{n-1} or a_{n+2}.

EXAMPLE 5 Writing a Recursive Rule

Write a recursive rule for the sequence.

−30, −18, −6, 6, 18, . . .

SOLUTION

Use a table to organize the terms and find the pattern.

Position, *n*	1	2	3	4	5
Term, a_n	−30	−18	−6	6	18

+12 +12 +12 +12

The sequence is arithmetic, with first term $a_1 = -30$ and common difference $d = 12$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 12$ Substitute 12 for d.

▶ So, a recursive rule for the sequence is $a_1 = -30$, $a_n = a_{n-1} + 12$.

COMMON ERROR

When writing a recursive rule for a sequence, you need to write both the beginning term(s) and the recursive equation.

SELF-ASSESSMENT

1 I do not understand. **2** I can do it with help. **3** I can do it on my own. **4** I can teach someone else.

15. WHICH ONE DOESN'T BELONG? Which rule does *not* belong with the other three? Explain your reasoning.

$a_1 = -1, a_n = a_{n-1} + 5$

$a_n = 6n - 2$

$a_1 = -3, a_n = a_{n-1} + 1$

$a_1 = 9, a_n = a_{n-1} + 4$

Write a recursive rule for the sequence.

16. 8, 3, −2, −7, −12, . . .

17. 1.3, 2.6, 3.9, 5.2, 6.5, . . .

18. Write a recursive rule for the height of the sunflower over time.

Translating between Recursive and Explicit Rules

EXAMPLE 6 Translating from a Recursive Rule to an Explicit Rule

WATCH

Write an explicit rule for the recursive rule.

$$a_1 = 25, a_n = a_{n-1} - 10$$

SOLUTION

The recursive rule represents an arithmetic sequence, with first term $a_1 = 25$ and common difference $d = -10$.

$a_n = a_1 + (n - 1)d$	Explicit rule for an arithmetic sequence
$a_n = 25 + (n - 1)(-10)$	Substitute 25 for a_1 and -10 for d.
$a_n = -10n + 35$	Simplify.

▶ An explicit rule for the sequence is

$$a_n = -10n + 35.$$

EXAMPLE 7 Translating from an Explicit Rule to a Recursive Rule

WATCH

Write a recursive rule for the explicit rule.

$$a_n = -2n + 3$$

SOLUTION

The explicit rule represents an arithmetic sequence, with first term $a_1 = -2(1) + 3 = 1$ and common difference $d = -2$.

$a_n = a_{n-1} + d$	Recursive equation for an arithmetic sequence
$a_n = a_{n-1} + (-2)$	Substitute -2 for d.
$a_n = a_{n-1} - 2$	Simplify.

▶ So, a recursive rule for the sequence is

$$a_1 = 1, a_n = a_{n-1} - 2.$$

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Write an explicit rule for the recursive rule.

19. $a_1 = -45, a_n = a_{n-1} + 20$

20. $a_1 = 13, a_{n-1} - 3$

Write a recursive rule for the explicit rule.

21. $a_n = -n + 1$

22. $a_n = 4n - 1.5$

23. **WRITING** Explain the difference between an explicit rule and a recursive rule.

2.3 Practice WITH CalcChat® AND CalcView®

In Exercises 1 and 2, write the next three terms of the arithmetic sequence.

1. First term: 2
Common difference: 13

2. First term: 18
Common difference: −6

In Exercises 3–6, find the common difference of the arithmetic sequence.

3. 13, 18, 23, 28, . . . **4.** 175, 150, 125, 100, . . .

5. 4, $3\frac{2}{3}$, $3\frac{1}{3}$, 3, . . . **6.** 6.5, 5, 3.5, 2, . . .

In Exercises 7–12, write the next three terms of the arithmetic sequence. *Example 1*

7. 19, 22, 25, 28, . . . **8.** 1, 12, 23, 34, . . .

9. 16, 21, 26, 31, . . . **10.** 60, 30, 0, −30, . . .

11. 1.3, 1, 0.7, 0.4, . . . **12.** $\frac{5}{6}$, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, . . .

In Exercises 13–18, graph the arithmetic sequence. *Example 2*

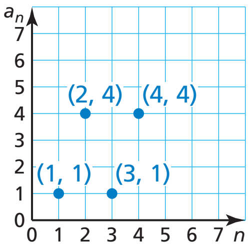
13. 4, 12, 20, 28, . . . **14.** −15, 0, 15, 30, . . .

15. −1, −3, −5, −7, . . . **16.** 2, 19, 36, 53, . . .

17. 0, $4\frac{1}{2}$, 9, $13\frac{1}{2}$, . . . **18.** 6, 5.25, 4.5, 3.75, . . .

In Exercises 19–22, determine whether the graph represents an arithmetic sequence. Explain. *Example 3*

19.



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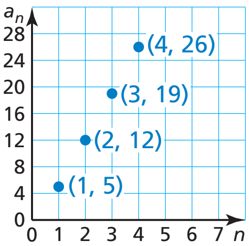

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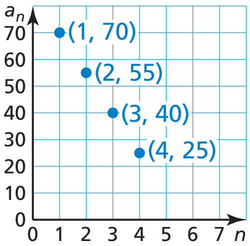

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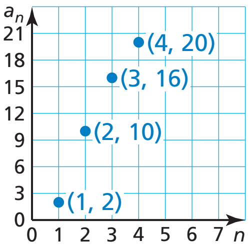
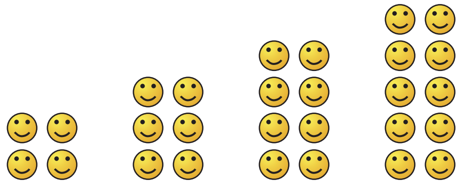


In Exercises 23–26, determine whether the sequence is arithmetic. If so, find the common difference.

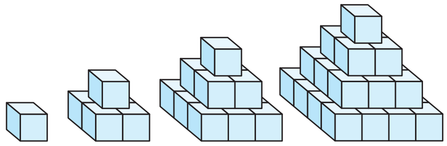
23. 13, 26, 39, 52, . . . **24.** 5, 9, 14, 20, . . .

25. 48, 24, 12, 6, . . . **26.** 87, 81, 75, 69, . . .

27. **MP PATTERNS** Write a sequence that represents the number of smiley faces in each group. Is the sequence arithmetic? Explain.



28. **MP PATTERNS** Write a sequence that represents the number of cubes in each group. Is the sequence arithmetic? Explain.



In Exercises 29–34, write an equation for the *n*th term of the arithmetic sequence. Then find a_{10}. *Example 4*

29. −5, −4, −3, −2, . . .

30. −6, −9, −12, −15, . . .

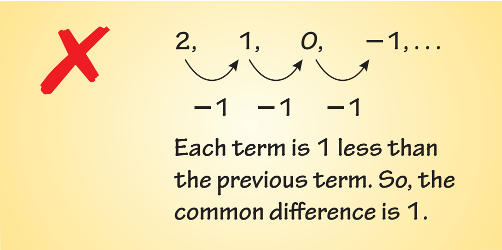
31. $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, . . .

32. 100, 110, 120, 130, . . .

33. 10, 0, −10, −20, . . .

34. $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, . . .

35. **ERROR ANALYSIS** Describe and correct the error in finding the common difference of the arithmetic sequence.



2, 1, 0, −1, . . .
−1 −1 −1

Each term is 1 less than the previous term. So, the common difference is 1.

36. **ERROR ANALYSIS** Describe and correct the error in writing an equation for the *n*th term of the arithmetic sequence.

✗

14, 22, 30, 38, . . .

$a_n = a_1 + nd$

$a_n = 14 + 8n$

In Exercises 37 and 38, write the first six terms of the sequence. Then graph the sequence.

37. $a_1 = 0, a_n = a_{n-1} + 2$

38. $a_1 = 10, a_n = a_{n-1} - 5$

In Exercises 39–44, write a recursive rule for the sequence. ▶ *Example 5*

39.

n	1	2	3	4
a_n	7	16	25	34

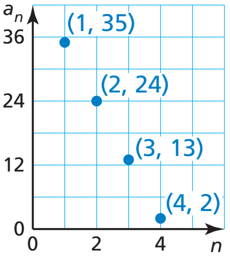
40.

n	1	2	3	4
a_n	−2	1	4	7

41. 3, 11, 19, 27, 35, . . .

42. 0, −3, −6, −9, −12, . . .

43.



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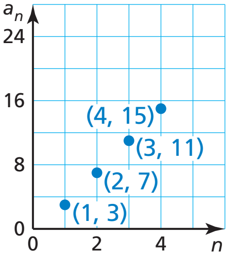


45. **MODELING REAL LIFE** Write a recursive rule that represents the height of a redwood tree over time.

Year	Height (feet)
1	51
2	54
3	57
4	60

46. **MODELING REAL LIFE** Write a recursive rule that represents the length of the deer antler over time.

Day	Length (inches)
1	$4\frac{1}{2}$
2	$4\frac{3}{4}$
3	5
4	$5\frac{1}{4}$

In Exercises 47–50, write an explicit rule for the recursive rule. ▶ *Example 6*

47. $a_1 = -3, a_n = a_{n-1} + 3$

48. $a_1 = 8, a_n = a_{n-1} - 12$

49. $a_1 = 4, a_n = a_{n-1} + 1.75$

50. $a_1 = 16, a_n = a_{n-1} + 0.5$

In Exercises 51–54, write a recursive rule for the explicit rule. ▶ *Example 7*

51. $a_n = 6n - 20$

52. $a_n = -4n + 2$

53. $a_n = 9n - 8$

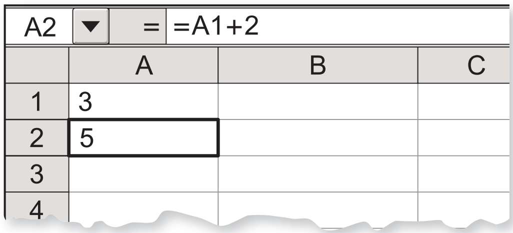
54. $a_n = 0.2n + 2.8$

In Exercises 55 and 56, graph the first four terms of the sequence with the given description. Write a recursive rule and an explicit rule for the sequence.

55. The first term of a sequence is 5. Each term of the sequence is 15 more than the preceding term.

56. The first term of a sequence is 19. Each term of the sequence is 13 less than the preceding term.

57. **MP** **USING TOOLS** You can use a spreadsheet to generate the terms of a sequence.



A2 ▼ = =A1+2

	A	B	C
1	3		
2	5		
3			
4			

To generate the terms of the sequence $a_1 = 3$, $a_n = a_{n-1} + 2$, enter the value of a_1, 3, into cell A1. Then enter "=A1+2" into cell A2, as shown. Use the *fill down* feature to generate the first 10 terms of the sequence. Repeat this process several times with different arithmetic sequences.

58. **MP** **REASONING** The explicit rule $a_n = a_1 + (n - 1)d$ defines an arithmetic sequence.

a. Explain why $a_{n-1} = a_1 + [(n - 1) - 1]d$.

b. Show that a recursive equation for the sequence is $a_n = a_{n-1} + d$.

59. **MP** **NUMBER SENSE** The fifth term of an arithmetic sequence is 21. The common difference of the sequence is 1.5 times the first term. Graph the sequence.

60. **MP** **NUMBER SENSE** The fourth term of an arithmetic sequence is 20. The common difference of the sequence is $-\frac{1}{5}$ times the first term. Graph the sequence.

61. **MODELING REAL LIFE** The total number of babies born in a country each minute after midnight on January 1st can be estimated by the arithmetic sequence shown in the table.

Minutes after midnight January 1st	1	2	3	4
Total babies born	5	10	15	20

a. Write a function that represents the sequence. Then graph the function.

b. Estimate how many minutes after midnight on January 1st it takes for 100 babies to be born.

62. **MODELING REAL LIFE** The gross revenue from a musical each week after opening night can be approximated by the arithmetic sequence shown in the table.

Week	1	2	3	4
Gross revenue (millions of dollars)	2.6	2.4	2.2	2.0

a. Write a function that represents the sequence. Then graph the function.

b. In what week does the musical earn $1.6 million?

63. **COLLEGE PREP** Which function represents the sequence 5, 1.5, −2, −5.5, . . .?

Ⓐ $f(n) = -3.5n + 5$

Ⓑ $g(n) = -3.5n + 8.5$

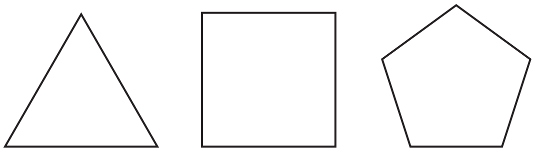
Ⓒ $h(n) = 3.5n + 1.5$

Ⓓ $k(n) = 5n - 8.5$

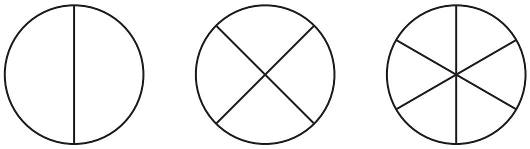
64. **OPEN-ENDED** Write the first four terms of two different arithmetic sequences with a common difference of −3. Write an equation for the *n*th term of each sequence.

MP **REPEATED REASONING** **In Exercises 65 and 66, a sequence represents the figures. (a) Draw the next three figures represented by the sequence and (b) describe the figure represented by the 20th number in the sequence.**

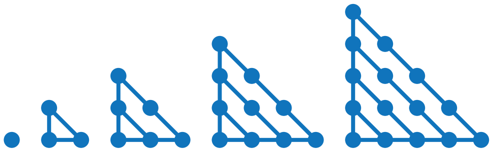
65.



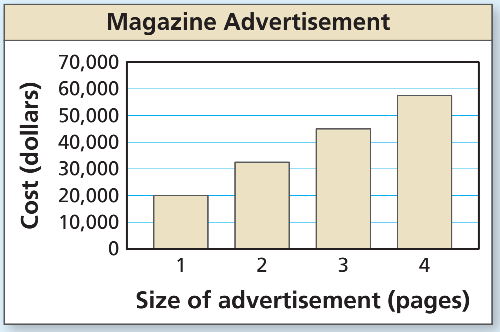
66.



67. **CRITICAL THINKING** Your friend says that the figures shown cannot be represented by an arithmetic sequence. Describe a sequence that supports your friend's claim. Then describe a sequence that does *not* support your friend's claim.



68. **HOW DO YOU SEE IT?**
The bar graph shows the costs of advertising in a magazine.

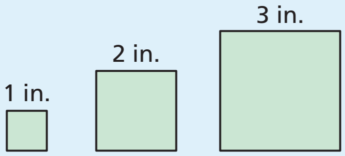


a. Does the graph represent an arithmetic sequence? Explain.

b. Explain how you would estimate the cost of a six-page advertisement in the magazine.

69. **MP** **REPEATED REASONING** Firewood is stacked in a pile. The bottom row has 20 logs, and the top row has 14 logs. Each row has one more log than the row above it. How many logs are in the pile?

70. **THOUGHT PROVOKING**
Write a function that you can use to find the increase in area from one square to the next as the side length increases by 1 inch. Explain your reasoning. Then find the increase in area from the 10th square to the 11th square.



71. **MP** **PROBLEM SOLVING** A train stops at a station every 12 minutes starting at 6:00 A.M. You arrive at the station at 7:29 A.M. How long must you wait for the train?

72. **MAKING AN ARGUMENT** Can a function with a range of all real numbers greater than or equal to zero represent an arithmetic sequence? Explain your reasoning.

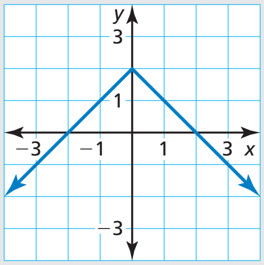
73. **DIG DEEPER** Let x be a constant. Determine whether each sequence is an arithmetic sequence. Explain.

a. $x + 6, 3x + 6, 5x + 6, 7x + 6, \ldots$

b. $x + 1, 3x + 1, 9x + 1, 27x + 1, \ldots$

REVIEW & REFRESH

74. Does the graph represent a *linear* or *nonlinear* function? Explain.

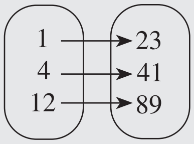


In Exercises 75 and 76, use the graphs of f and g to describe the transformations from the graph of f to the graph of g.

75. $f(x) = -2x + 7;\ g(x) = -10x + 5$

76. $f(x) = -\frac{3}{4}x + 6;\ g(x) = \frac{3}{4}x - 3$

77. **MP** **REASONING** Write a function f that represents the arithmetic sequence shown in the mapping diagram.



78. **MODELING REAL LIFE** You can use at most 5 gigabytes of data per month on your cell phone. Your data usage so far for the month is 1.24 gigabytes. What are the possible amounts of data you can use for the remainder of the month?

In Exercises 79–82, classify the number as *rational* or *irrational*. Explain your reasoning.

79. $\frac{9}{13}$

80. $\sqrt{7}$

81. $-\sqrt{49}$

82. 2π

In Exercises 83 and 84, use the graphs of f and g to describe the transformation from the graph of f to the graph of g.

83. $f(x) = -\frac{1}{2}x;\ g(x) = f(x + 2)$

84. $f(x) = 3x - 1;\ g(x) = -f(x)$

85. **MODELING REAL LIFE** You have $15 to purchase pecans and walnuts. The equation $12x + 7.5y = 15$ models this situation, where x is the number of pounds of pecans and y is the number of pounds of walnuts.

a. Interpret the terms and coefficients in the equation.

b. Graph the equation. Interpret the intercepts.

In Exercises 86 and 87, determine whether the sequence is arithmetic. If so, find the common difference.

86. $27, 9, 3, 1, \ldots$

87. $-14, -9, -4, 1, \ldots$