5.8 Coordinate Proofs

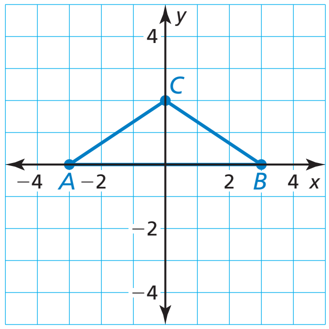
Learning Target: Use coordinates to write proofs.

Success Criteria:
- I can place figures in a coordinate plane.
- I can write plans for coordinate proofs.
- I can write coordinate proofs.

EXPLORE IT! Writing a Proof Using Coordinates

Work with a partner.

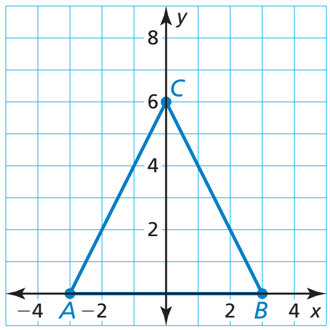
a. Draw $\overline{AB}$ with endpoints $A(-3, 0)$ and $B(3, 0)$. Then draw $\triangle ABC$ so that C lies on the y-axis.

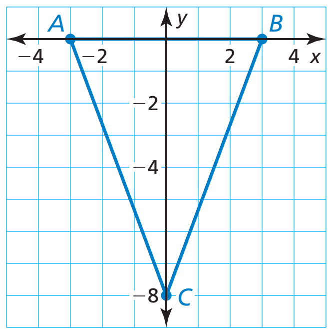


b. Classify $\triangle ABC$ by its sides.

5 MTR USE STRUCTURE
How can you write coordinates that represent any point on the y-axis?

c. Repeat parts (a) and (b), drawing C at different points on the y-axis. What do you notice?





d. How can you prove that if C lies on the y-axis, then $\triangle ABC$ is an isosceles triangle?

e. What coordinates of C make $\triangle ABC$ an equilateral triangle?

Geometric Reasoning
MA.912.GR.3.2 Given a mathematical context, use coordinate geometry to classify or justify definitions, properties and theorems involving circles, triangles or quadrilaterals.

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Placing Figures in a Coordinate Plane

Vocabulary

coordinate proof, *p. 280*

A **coordinate proof** involves placing geometric figures in a coordinate plane. When you use variables to represent the coordinates of a figure in a coordinate proof, the results are true for all figures of that type.

EXAMPLE 1 Placing a Figure in a Coordinate Plane

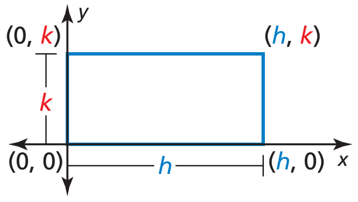
Place each figure in a coordinate plane in a way that is convenient for finding side lengths. Assign coordinates to each vertex.

a. a rectangle **b.** a triangle

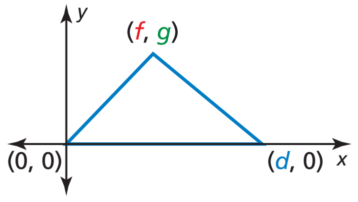
SOLUTION

It is easy to find lengths of horizontal and vertical segments and distances from (0, 0), so place one vertex at the origin and one or more sides on an axis.

a. Let h represent the length and k represent the width.



b. Notice that you need to use three different variables.

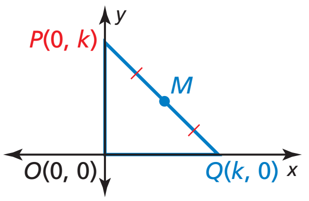


EXAMPLE 2 Applying Variable Coordinates

Place an isosceles right triangle in a coordinate plane. Then find the length of the hypotenuse and the coordinates of its midpoint M.

SOLUTION

Place $\triangle PQO$ with the right angle at the origin. Let the length of the legs be k. Then the vertices are located at $P(0, k)$, $Q(k, 0)$, and $O(0, 0)$.



Use the Distance Formula to find PQ, the length of the hypotenuse.

$$\begin{aligned} PQ &= \sqrt{(k-0)^2 + (0-k)^2} \\ &= \sqrt{k^2 + (-k)^2} \\ &= \sqrt{k^2 + k^2} \\ &= \sqrt{2k^2} \\ &= k\sqrt{2} \end{aligned}$$

Use the Midpoint Formula to find the midpoint M of the hypotenuse.

$$M\left(\frac{0+k}{2}, \frac{k+0}{2}\right) = M\left(\frac{k}{2}, \frac{k}{2}\right)$$

▶ So, the length of the hypotenuse is $k\sqrt{2}$ and the midpoint of the hypotenuse is $\left(\frac{k}{2}, \frac{k}{2}\right)$.

2 MTR USE ANOTHER METHOD

Show how to solve Example 2 by drawing the hypotenuse on the x-axis and a vertex on the y-axis.

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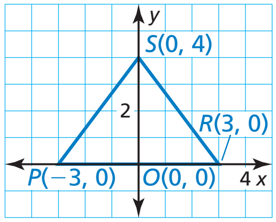
EXAMPLE 3 Writing a Plan for a Coordinate Proof

WATCH

Write a plan to prove that $\overrightarrow{SO}$ bisects $\angle PSR$.



Given Coordinates of vertices of $\triangle POS$ and $\triangle ROS$

Prove $\overrightarrow{SO}$ bisects $\angle PSR$.

SOLUTION

Plan for Proof Use the Distance Formula to find the side lengths of $\triangle POS$ and $\triangle ROS$. Then use the SSS Congruence Theorem to show that $\triangle POS \cong \triangle ROS$. Finally, use the fact that corresponding parts of congruent triangles are congruent to conclude that $\angle PSO \cong \angle RSO$, which implies that $\overrightarrow{SO}$ bisects $\angle PSR$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. Show another way to place the rectangle in Example 1 part (a) that is convenient for finding side lengths. Assign new coordinates.

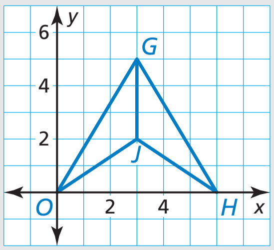
2. A square has vertices (0, 0), (m, 0), and (0, m). Find the fourth vertex.

3. Graph the points $O(0, 0)$, $H(m, n)$, and $J(m, 0)$. What kind of triangle is $\triangle OHJ$? Find the side lengths and the coordinates of the midpoint of each side.

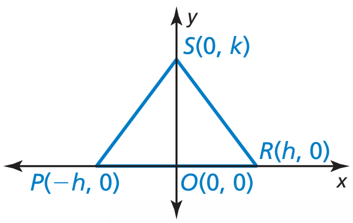
4. Write a plan for the proof.

Given $\overrightarrow{GJ}$ bisects $\angle OGH$.

Prove $\triangle GJO \cong \triangle GJH$



The coordinate proof in Example 3 applies to a specific triangle. When you want to prove a statement about a more general set of figures, it is helpful to use variables as coordinates.



For instance, you can use the variable coordinates shown at the left to duplicate the proof in Example 3. Once this is done, you can conclude that $\overrightarrow{SO}$ bisects $\angle PSR$ for any triangle whose coordinates fit the given pattern.

When writing a coordinate proof, you may not be given the coordinates of a figure. You may have to place the figure in the coordinate plane. Once the figure is placed in a coordinate plane, you may be able to prove statements about the figure.

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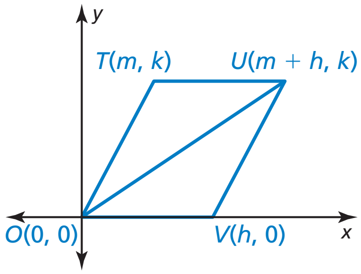
Writing Coordinate Proofs

EXAMPLE 4 Writing a Coordinate Proof

WATCH

Write a coordinate proof.



Given Coordinates of vertices of quadrilateral *OTUV*

Prove $\triangle OTU \cong \triangle UVO$

SOLUTION

Segments $\overline{OV}$ and $\overline{UT}$ have the same length.

$$OV = |h - 0| = h$$

$$UT = |(m + h) - m| = h$$

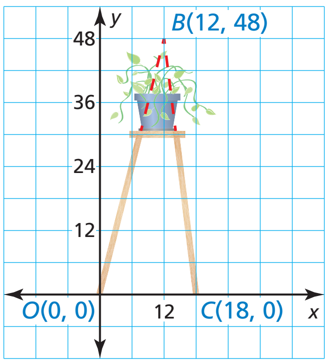
Horizontal segments $\overline{UT}$ and $\overline{OV}$ each have a slope of 0, which implies that they are parallel. Segment $\overline{OU}$ intersects $\overline{UT}$ and $\overline{OV}$ to form congruent alternate interior angles, $\angle TUO$ and $\angle VOU$. By the Reflexive Property of Segment Congruence, $\overline{OU} \cong \overline{OU}$.

▶ So, you can apply the SAS Congruence Theorem to conclude that $\triangle OTU \cong \triangle UVO$.

7 MTR

EXAMPLE 5 Modeling Real Life

WATCH



You buy a tall, four-legged plant stand. When you place a plant on the stand, the stand appears to be unstable under the weight of the plant. The diagram at the left shows a coordinate plane superimposed on one pair of the plant stand's legs. One unit in the coordinate plane represents one inch. The legs are extended to form $\triangle OBC$. Prove that $\triangle OBC$ is a scalene triangle. Explain why the plant stand may be unstable.

SOLUTION

First, find the side lengths of $\triangle OBC$.

$$OB = \sqrt{(12 - 0)^2 + (48 - 0)^2} = \sqrt{2448} \approx 49.5 \text{ in.}$$

$$BC = \sqrt{(18 - 12)^2 + (0 - 48)^2} = \sqrt{2340} \approx 48.4 \text{ in.}$$

$$OC = |18 - 0| = 18 \text{ in.}$$

▶ Because $\triangle OBC$ has no congruent sides, $\triangle OBC$ is a scalene triangle by definition. The plant stand may be unstable because $\overline{OB}$ is longer than $\overline{BC}$, so the plant stand is leaning to the right.

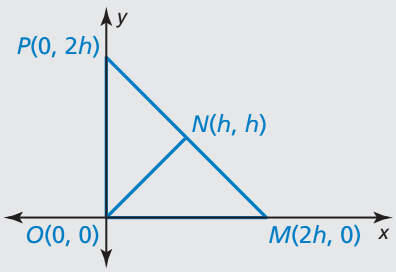
SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

5. Write a coordinate proof.

Given Coordinates of vertices of $\triangle NPO$ and $\triangle NMO$

Prove $\triangle NPO \cong \triangle NMO$



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5.8 Practice WITH CalcChat® AND CalcView®

In Exercises 1–4, place the figure in a coordinate plane in a convenient way. Assign coordinates to each vertex. Explain the advantages of your placement. *(See Example 1.)*

▶ **1.** a right triangle with leg lengths of 3 units and 2 units

2. a square with a side length of 3 units

3. an isosceles right triangle with leg length p

4. a scalene triangle with one side length of $2m$

In Exercises 5–8, place the figure in a coordinate plane and find the indicated length. *(See Example 2.)*

▶ **5.** a right triangle with leg lengths of 7 and 9 units; Find the length of the hypotenuse.

6. an isosceles triangle with a base length of 60 units and a height of 50 units; Find the length of one of the legs.

7. a rectangle with a length of 5 units and a width of 4 units; Find the length of the diagonal.

8. a square with side length n; Find the length of the diagonal.

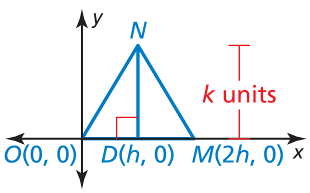
In Exercises 9 and 10, graph the triangle with the given vertices. Find the length and the slope of each side of the triangle. Then find the coordinates of the midpoint of each side. Is the triangle a right triangle? isosceles? Explain. (Assume all variables are positive and $m \neq n$.)

▶ **9.** $A(0, 0)$, $B(h, h)$, $C(2h, 0)$

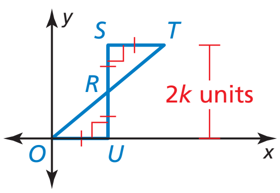
10. $D(0, n)$, $E(m, n)$, $F(m, 0)$

In Exercises 11 and 12, find the coordinates of any unlabeled vertices. Then find the indicated length(s).

11. Find ON and MN.



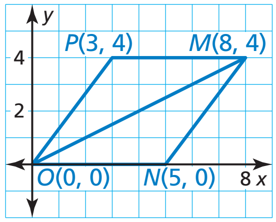
12. Find OT.



In Exercises 13 and 14, write a plan for the proof. *(See Example 3.)*

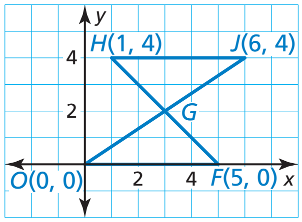
▶ **13.** **Given** Coordinates of vertices of $\triangle OPM$ and $\triangle ONM$

Prove $\triangle OPM$ and $\triangle ONM$ are isosceles triangles.



14. **Given** G is the midpoint of $\overline{HF}$.

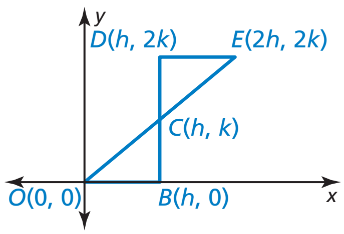
Prove $\triangle GHJ \cong \triangle GFO$



PROOF In Exercises 15 and 16, write a coordinate proof. *(See Example 4.)*

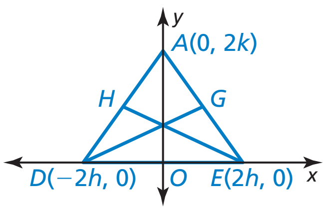
▶ **15.** **Given** Coordinates of vertices of $\triangle DEC$ and $\triangle BOC$

Prove $\triangle DEC \cong \triangle BOC$



16. **Given** Coordinates of vertices of $\triangle DEA$, H is the midpoint of $\overline{DA}$, G is the midpoint of $\overline{EA}$.

Prove $\overline{DG} \cong \overline{EH}$



17. **MODELING REAL LIFE** A manufacturer cuts a piece of metal for a microscope. The resulting piece of metal can be represented in a coordinate plane by a triangle with $A(0, 0)$, $B(5, 12)$, and $C(10, 0)$. One unit in the coordinate plane represents one millimeter. Prove that $\triangle ABC$ is isosceles. *(See Example 5.)*

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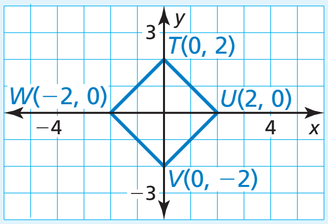
7 MTR **18. MODELING REAL LIFE** You design the front of a phone tripod using a coordinate plane on a computer program. The coordinates of the vertices of the triangle at the front of the tripod are $A(0, 0)$, $B(12, 16)$, and $C(22, 0)$. One unit in the coordinate plane represents one inch. Prove that $\triangle ABC$ is a scalene triangle. Describe how to adjust point C to improve stability.

4 MTR **19. MAKING AN ARGUMENT** Your friend says that quadrilateral $PQRS$ with vertices $P(0, 2)$, $Q(3, -4)$, $R(1, -5)$, and $S(-2, 1)$ is a rectangle. Is your friend correct? Explain.

20. THOUGHT PROVOKING
Choose one of the theorems you have encountered that is easier to prove with a coordinate proof than with another type of proof. Explain. Then write a coordinate proof.

5 MTR **21. STRUCTURE** Write an algebraic expression for the coordinates of each endpoint of a line segment whose midpoint is the origin.

22. HOW DO YOU SEE IT?
Without performing any calculations, how do you know that the diagonals of square $TUVW$ are perpendicular to each other?



23. REASONING A rectangle has a length of ℓ inches and width of w inches. Represent this rectangle in the coordinate plane. Show that the diagonals of the rectangle are congruent.

24. PROOF Write a coordinate proof for each statement.

a. The midpoint of the hypotenuse of a right triangle is the same distance from each vertex of the triangle.

b. Any two congruent right isosceles triangles can be combined to form a single isosceles triangle.

REVIEW & REFRESH

WATCH

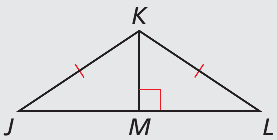
In Exercises 25 and 26, solve the equation. Justify each step.

25. $6x + 13 = -5$

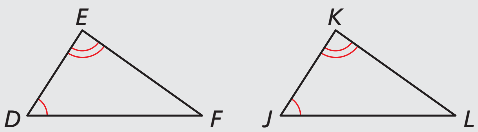
26. $3(x - 1) = -(x + 10)$

27. Factor the polynomial $14a^2 + 23a + 3$.

28. Explain how to prove that $\angle J \cong \angle L$.



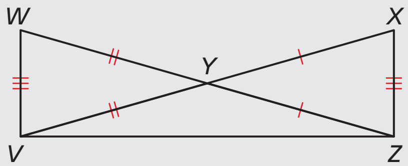
29. Decide whether enough information is given to prove that the $\triangle DEF$ and $\triangle JKL$ are congruent. If so, state the theorem you can use.



30. Write a proof.

Given $\overline{XY} \cong \overline{ZY}$, $\overline{WY} \cong \overline{VY}$, $\overline{VW} \cong \overline{ZX}$

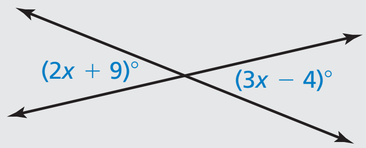
Prove $\triangle VWZ \cong \triangle ZXV$



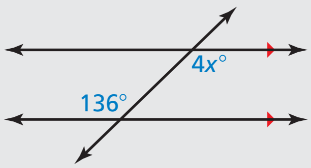
31. Place a square with a side length of m units in a coordinate plane in a convenient way for finding the length of the diagonal. Assign coordinates to each vertex.

In Exercises 32 and 33, find the value of x.

32.



33.



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