6.5 Properties of Logarithms

Learning Target: Use properties of logarithms.

Success Criteria:
- I can evaluate logarithms.
- I can expand or condense logarithmic expressions.
- I can explain how to use the change-of-base formula.

EXPLORE IT! Deriving Properties of Logarithms

Work with a partner. You can use properties of exponents to derive several properties of logarithms. Let $x = \log_b m$ and $y = \log_b n$. The corresponding exponential forms of these two equations are

$$b^x = m \qquad \text{and} \qquad b^y = n.$$

a. The diagram shows a way to derive the Product Property of Logarithms. Complete and explain the diagram.

Exponential Form of *mn*

$mn = b^x b^y$

↓

$mn = b^{x+y}$ ⟶ $\log_b mn = x + y$

Logarithmic Form of $mn = b^{x+y}$

↓

Product Property of Logarithms

$\log_b mn =$ ______

> **4 MTR CONSTRUCT AN ARGUMENT**
> Can you extend the Product Property of Logarithms to more than two factors? Explain.

b. Derive the Quotient Property of Logarithms shown below using a diagram similar to the one in part (a). Explain your reasoning.

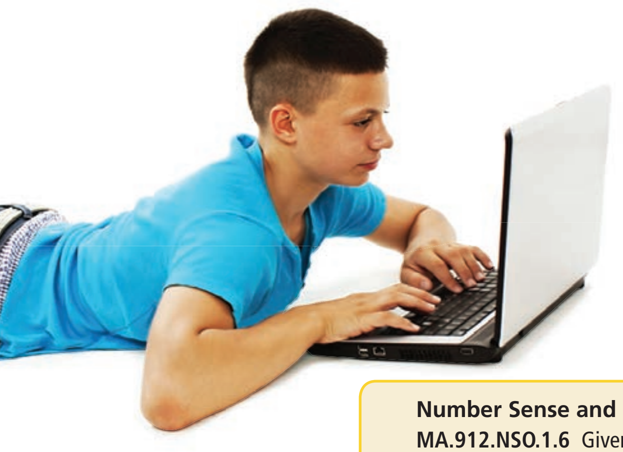
$$\log_b \frac{m}{n} = \log_b m - \log_b n$$ Quotient Property of Logarithms

Give some examples to show that the property works. Revise your work if needed.

c. Use the substitution $m = b^x$ to derive the Power Property of Logarithms shown below.

$$\log_b m^n = n \log_b m$$ Power Property of Logarithms

d. How are these three properties of logarithms similar to properties of exponents?



Number Sense and Operations

MA.912.NSO.1.6 Given a numerical logarithmic expression, evaluate and generate equivalent numerical expressions using the properties of logarithms or exponents.

MA.912.NSO.1.7 Given an algebraic logarithmic expression, generate an equivalent algebraic expression using the properties of logarithms or exponents.

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Properties of Logarithms

You know that the logarithmic function with base b is the inverse function of the exponential function with base b. Because of this relationship, it makes sense that logarithms have properties similar to properties of exponents.

KEY IDEA

Properties of Logarithms

Let b, m, and n be positive real numbers with $b \neq 1$.

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

STUDY TIP

These three properties of logarithms correspond to these three properties of exponents.

$a^m a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

EXAMPLE 1 Using Properties of Logarithms

WATCH

Use $\log_2 3 \approx 1.585$ and $\log_2 7 \approx 2.807$ to evaluate each logarithm.

a. $\log_2 \frac{3}{7}$ **b.** $\log_2 21$ **c.** $\log_2 49$

SOLUTION

a. $\log_2 \frac{3}{7} = \log_2 3 - \log_2 7$ Quotient Property

$\approx 1.585 - 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= -1.222$ Subtract.

b. $\log_2 21 = \log_2(3 \cdot 7)$ Write 21 as $3 \cdot 7$.

$= \log_2 3 + \log_2 7$ Product Property

$\approx 1.585 + 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= 4.392$ Add.

c. $\log_2 49 = \log_2 7^2$ Write 49 as 7^2.

$= 2 \log_2 7$ Power Property

$\approx 2(2.807)$ Use the given value of $\log_2 7$.

$= 5.614$ Multiply.

COMMON ERROR

Note that in general,

$\log_b \frac{m}{n} \neq \frac{\log_b m}{\log_b n}$ and

$\log_b mn \neq (\log_b m)(\log_b n)$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Use $\log_6 5 \approx 0.898$ and $\log_6 8 \approx 1.161$ to evaluate the logarithm.

1. $\log_6 \frac{5}{8}$ **2.** $\log_6 40$

3. $\log_6 64$ **4.** $\log_6 125$

5 MTR **5.** **STRUCTURE** Without using technology, can you use the approximations given below to evaluate $\ln x$ for all integer values of x between 1 and 10? Explain your reasoning.

$\ln 2 \approx 0.6931, \quad \ln 3 \approx 1.0986, \quad \ln 5 \approx 1.6094$

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Expanding and Condensing Logarithmic Expressions

EXAMPLE 2 Expanding a Logarithmic Expression WATCH

Expand $\ln \frac{5x^7}{y}$.

STUDY TIP

When you are expanding or condensing an expression involving logarithms, you can assume that any variables are positive.

SOLUTION

$$\ln \frac{5x^7}{y} = \ln 5x^7 - \ln y \qquad \text{Quotient Property}$$

$$= \ln 5 + \ln x^7 - \ln y \qquad \text{Product Property}$$

$$= \ln 5 + 7 \ln x - \ln y \qquad \text{Power Property}$$

EXAMPLE 3 Condensing a Logarithmic Expression WATCH

Condense $\log 9 + 3 \log 2 - \log 3$.

SOLUTION

$$\log 9 + 3 \log 2 - \log 3 = \log 9 + \log 2^3 - \log 3 \qquad \text{Power Property}$$

$$= \log(9 \cdot 2^3) - \log 3 \qquad \text{Product Property}$$

$$= \log \frac{9 \cdot 2^3}{3} \qquad \text{Quotient Property}$$

$$= \log 24 \qquad \text{Simplify.}$$

SELF-ASSESSMENT 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Expand the logarithmic expression.

6. $\log_6 3x^4$

7. $\ln \frac{5}{12x}$

8. $\log_5 2\sqrt{x}$

9. **REASONING** Which property of logarithms do you need to use to condense the expression $\log_3 2x + \log_3 y$?

Condense the logarithmic expression.

10. $\log x - \log 9$

11. $4 \ln x + 8 \ln y$

12. $\ln 4 + 3 \ln 3 - \ln 12$

Change-of-Base Formula

Logarithms with any base other than 10 or e can be written in terms of common or natural logarithms using the *change-of-base formula*. This allows you to evaluate any logarithm using a calculator.

KEY IDEA

Change-of-Base Formula

If a, b, and c are positive real numbers with $b \neq 1$ and $c \neq 1$, then

$$\log_c a = \frac{\log_b a}{\log_b c}.$$

In particular, $\log_c a = \frac{\log a}{\log c}$ and $\log_c a = \frac{\ln a}{\ln c}$.

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EXAMPLE 4 Changing a Base Using Common Logarithms

WATCH

Evaluate $\log_3 8$ using common logarithms.

SOLUTION

$$\log_3 8 = \frac{\log 8}{\log 3} \qquad \log_c a = \frac{\log a}{\log c}$$

$$\approx 1.893 \qquad \text{Use technology.}$$

ANOTHER WAY

You can also evaluate $\log_3 8$ using natural logarithms.

$$\log_3 8 = \frac{\ln 8}{\ln 3} \approx 1.893$$

Notice that you get the same answer whether you use natural logarithms or common logarithms in the change-of-base formula.

EXAMPLE 5 Changing a Base Using Natural Logarithms

WATCH

Evaluate $\log_6 24$ using natural logarithms.

SOLUTION

$$\log_6 24 = \frac{\ln 24}{\ln 6} \qquad \log_c a = \frac{\ln a}{\ln c}$$

$$\approx 1.774 \qquad \text{Use technology.}$$

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EXAMPLE 6 Modeling Real Life

WATCH

For a sound with intensity I (in watts per square meter), the loudness $L(I)$ of the sound (in decibels) is given by the function

$$L(I) = 10 \log \frac{I}{I_0}$$

where I_0 is the intensity of a barely audible sound (about 10^{-12} watt per square meter). An artist in a recording studio turns up the volume of a track so that the intensity of the sound doubles. By how many decibels does the loudness increase?

SOLUTION

Let I be the original intensity, so that $2I$ is the doubled intensity.

increase in loudness $= L(2I) - L(I)$ — Write an expression.

$= 10 \log \frac{2I}{I_0} - 10 \log \frac{I}{I_0}$ — Substitute.

$= 10\left(\log \frac{2I}{I_0} - \log \frac{I}{I_0}\right)$ — Distributive Property

$= 10\left(\log 2 + \log \frac{I}{I_0} - \log \frac{I}{I_0}\right)$ — Product Property

$= 10 \log 2$ — Simplify.

▶ The loudness increases by $10 \log 2$ decibels, or about 3 decibels.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Use the change-of-base formula to evaluate the logarithm.

13. $\log_5 8$ **14.** $\log_8 14$ **15.** $\log_{26} 9$ **16.** $\log_{12} 30$

17. REASONING Describe two ways to evaluate $\log_7 12$ using a calculator.

18. WHAT IF? In Example 6, the artist turns up the volume so that the intensity of the sound triples. By how many decibels does the loudness increase?

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6.5 Practice WITH CalcChat® AND CalcView®

In Exercises 1–4, match the expression with the logarithm that has the same value. Justify your answer.

1. $\log_3 6 - \log_3 2$ A. $\log_3 64$
2. $2 \log_3 6$ B. $\log_3 3$
3. $6 \log_3 2$ C. $\log_3 12$
4. $\log_3 6 + \log_3 2$ D. $\log_3 36$

In Exercises 5–10, use $\log_7 4 \approx 0.712$ and $\log_7 12 \approx 1.277$ to evaluate the logarithm. *(See Example 1.)*

▶ 5. $\log_7 3$ 6. $\log_7 48$

7. $\log_7 16$ 8. $\log_7 64$

9. $\log_7 \frac{1}{4}$ 10. $\log_7 \frac{1}{3}$

In Exercises 11–18, expand the logarithmic expression. *(See Example 2.)*

11. $\log_3 2x$ 12. $\log_8 3x$

▶ 13. $\log 10x^5$ 14. $\ln 3x^4$

15. $\ln \frac{x}{3y}$ 16. $\ln \frac{6x^2}{y^4}$

17. $\log_7 5\sqrt{x}$ 18. $\log_5 \sqrt[3]{x^2 y}$

4 MTR **ERROR ANALYSIS** In Exercises 19 and 20, describe and correct the error in expanding the logarithmic expression.

19. ✗ $\log_2 5x = (\log_2 5)(\log_2 x)$

20. ✗ $\ln 7x^3 = 3 \ln 7x$
$= 3(\ln 7 + \ln x)$
$= 3 \ln 7 + 3 \ln x$

In Exercises 21–28, condense the logarithmic expression. *(See Example 3.)*

21. $\log_4 7 - \log_4 10$ 22. $\ln 12 - \ln 4$

▶ 23. $6 \ln x + 4 \ln y$ 24. $2 \log x + \log 11$

25. $\log_5 4 + \frac{1}{3} \log_5 x$

26. $6 \ln 2 - 4 \ln y$

27. $5 \ln 2 + 7 \ln x + 4 \ln y$

28. $\log_3 4 + 2 \log_3 \frac{1}{2} + \log_3 x$

In Exercises 29–36, use the change-of-base formula to evaluate the logarithm. *(See Examples 4 and 5.)*

▶ 29. $\log_4 7$ 30. $\log_5 13$

31. $\log_9 15$ 32. $\log_8 22$

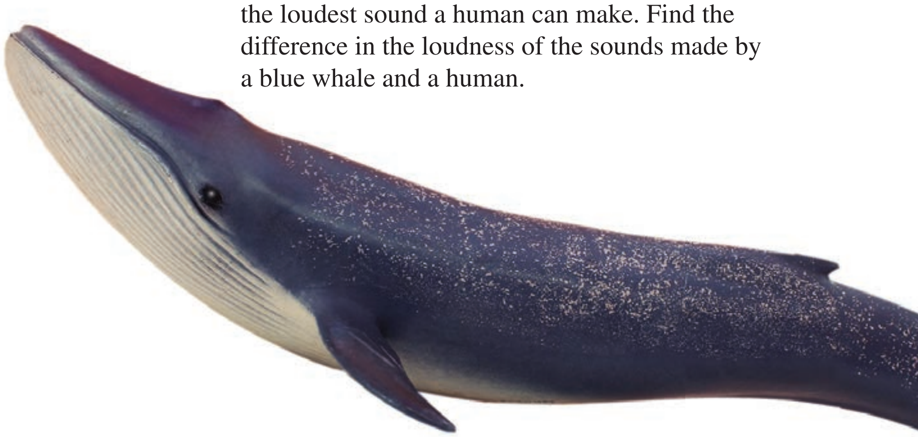
33. $\log_6 17$ 34. $\log_2 28$

35. $\log_7 \frac{3}{16}$ 36. $\log_3 \frac{9}{40}$

7 MTR **MODELING REAL LIFE** In Exercises 37 and 38, use the function $L(I) = 10 \log \frac{I}{I_0}$ given in Example 6.

37. The intensity of the sound of a television commercial is 10 times greater than the intensity of the television program it follows. By how many decibels does the loudness increase? *(See Example 6.)*

38. The blue whale can produce sound with an intensity that is 1 million times greater than the intensity of the loudest sound a human can make. Find the difference in the loudness of the sounds made by a blue whale and a human.



39. **COLLEGE PREP** Which of the following is *not* equivalent to $\log_5 \frac{y^4}{3x}$?

(A) $4 \log_5 y - \log_5 3x$

(B) $4 \log_5 y - \log_5 3 + \log_5 x$

(C) $4 \log_5 y - \log_5 3 - \log_5 x$

(D) $\log_5 y^4 - \log_5 3 - \log_5 x$

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40. **COLLEGE PREP** Which of the following equations is true?

Ⓐ $\log_7 x + 2\log_7 y = \log_7(x + y^2)$

Ⓑ $9\log x - 2\log y = \log \frac{x^9}{y^2}$

Ⓒ $5\log_4 x + 7\log_2 y = \log_6 x^5y^7$

Ⓓ $\log_9 x - 5\log_9 y = \log_9 \frac{x}{5y}$

41. **REASONING** Under certain conditions, the wind speed (in knots) at an altitude of h meters above a grassy plain can be modeled by the function $s(h) = 2\ln 100h$.

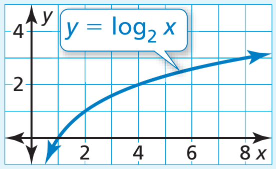
a. By what amount does the wind speed increase when the altitude doubles?

b. Show that the given function can be written in terms of common logarithms as

$$s(h) = \frac{2}{\log e}(\log h + 2).$$

42. **HOW DO YOU SEE IT?**

Use the graph to determine the value of $\frac{\log 8}{\log 2}$.



43. **REASONING** Determine whether $\log_b(M + N) = \log_b M + \log_b N$ is true for all positive, real values of M, N, and b (with $b \neq 1$). Justify your answer.

44. **THOUGHT PROVOKING**

Use properties of exponents to prove the change-of-base formula. (*Hint:* Let $x = \log_b a$, $y = \log_b c$, and $z = \log_c a$.)

45. **DIG DEEPER** Describe three ways to transform the graph of $f(x) = \log x$ to obtain the graph of $g(x) = \log 100x - 1$. Justify your answers.

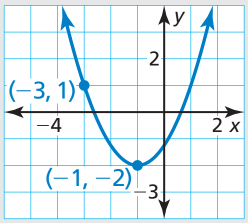
REVIEW & REFRESH

WATCH

In Exercises 46 and 47, rewrite the equation in exponential or logarithmic form.

46. $\log_4 1024 = 5$ **47.** $7^4 = 2401$

48. Write a quadratic function in standard form whose graph is shown.



49. Use the change-of-base formula to evaluate $\log_5 20$.

In Exercises 50 and 51, solve the equation by graphing.

50. $4x^2 - 3x - 6 = -x^2 + 5x + 3$

51. $-(x + 3)(x + 2) = x^2 - 6x$

52. **MODELING REAL LIFE** The wheels on your bicycle have a radius of 34 centimeters. You ride your bicycle 60 meters. How many complete revolutions does one wheel on your bicycle make?

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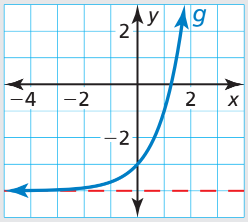
In Exercises 53 and 54, solve the inequality by graphing.

53. $x^2 + 13x + 42 < 0$ **54.** $-x^2 - 4x + 6 \leq -6$

55. Expand $\log \frac{y^3}{x^5}$.

56. The graph of g is a transformation of the graph of $f(x) = 3^x$. Write a rule for g.



In Exercises 57 and 58, perform the operation. Write the answer in standard form.

57. $(3 - i)(8 + 2i)$

58. $(6 + 11i) - (13 - 4i)$

In Exercises 59–62, simplify the expression.

59. $e^8 \bullet e^4$ **60.** $\frac{15e^3}{3e}$

61. $(5e^{4x})^3$ **62.** $\frac{e^{11} \bullet e^{-3}}{e^2}$

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