6.1 Exponential Growth and Decay Functions

Learning Target: Write and graph exponential growth and decay functions.

Success Criteria:
- I can identify and graph exponential growth and decay functions.
- I can write exponential growth and decay functions.
- I can solve real-life problems using exponential growth and decay functions.

EXPLORE IT! Describing Exponential Growth

Work with a partner. You are studying bacteria growth in a laboratory.

a. The study starts with a population of 100 bacteria. You notice that the population doubles every hour. Complete the table.

Time (hours), *t*	0	1	2	3	4	5
Population, *P*						

b. Write a model that represents the population P of the bacteria after t hours.

c. Use the model in part (b) to complete the table. By what factor does the population increase every half hour? Explain your reasoning.

Time (hours), *t*	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Population, *P*							

d. Use the model in part (b) to complete the table. By what factor does the population increase every 20 minutes? Explain your reasoning.

Time (hours), *t*	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
Population, *P*							

e. Create a graph that shows the population P after t hours. Would you expect the actual bacteria population to closely follow this model as t increases? Explain your reasoning.

f. The population P of a different type of bacteria after t hours can be represented by

$$P = 10(3)^t.$$

How does the growth pattern of this bacteria compare with the growth rate of the bacteria in parts (a)–(e)?

4 MTR CONSTRUCT AN ARGUMENT
Will the two bacteria populations ever be equal? Explain.

Algebraic Reasoning
MA.912.AR.5.4 Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.
MA.912.AR.5.7 Solve and graph mathematical and real-world problems that are modeled with exponential functions. Interpret key features and determine constraints in terms of the context.
Also MA.912.AR.1.1, MA.912.AR.5.5, MA.912.F.1.1, MA.912.FL.3.2

GO DIGITAL

Exponential Growth and Decay Functions

Vocabulary

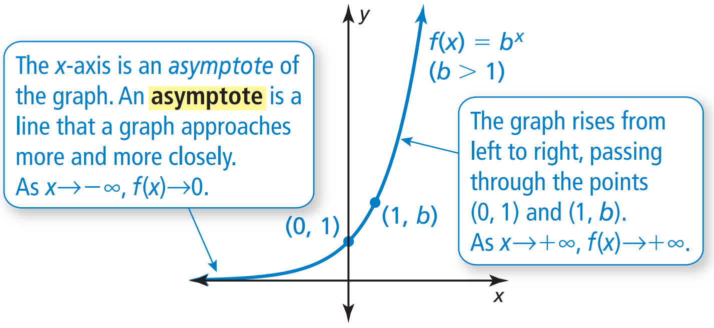
exponential function, *p. 314*
exponential growth function, *p. 314*
growth factor, *p. 314*
asymptote, *p. 314*
exponential decay function, *p. 314*
decay factor, *p. 314*

An **exponential function** has the form $y = ab^x$, where $a \neq 0$ and the base b is a positive real number other than 1. If $a > 0$ and $b > 1$, then $y = ab^x$ is an **exponential growth function**, and b is called the **growth factor**. The simplest type of exponential growth function has the form $y = b^x$.

KEY IDEA

Parent Function for Exponential Growth Functions

The function $f(x) = b^x$, where $b > 1$, is the parent function for the family of exponential growth functions with base b. The graph shows the general shape of an exponential growth function.



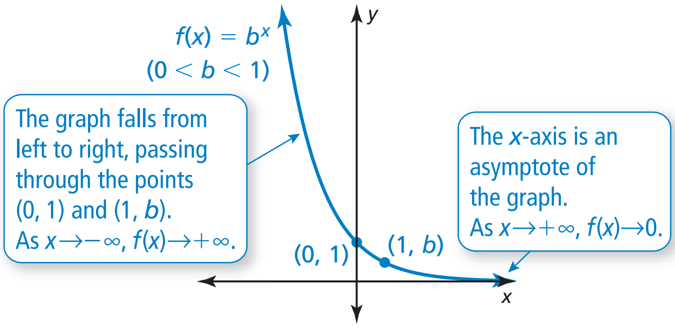
The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

If $a > 0$ and $0 < b < 1$, then $y = ab^x$ is an **exponential decay function**, and b is called the **decay factor**.

KEY IDEA

Parent Function for Exponential Decay Functions

The function $f(x) = b^x$, where $0 < b < 1$, is the parent function for the family of exponential decay functions with base b. The graph shows the general shape of an exponential decay function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

GO DIGITAL

EXAMPLE 1 Graphing Exponential Growth and Decay Functions

WATCH

Determine whether each function represents *exponential growth* or *exponential decay*. Then graph the function and describe the end behavior.

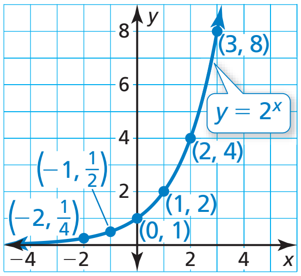
a. $y = 2^x$ **b.** $y = \left(\frac{1}{2}\right)^x$

SOLUTION

a. Step 1 Identify the value of the base. The base, 2, is greater than 1, so the function represents exponential growth.

Step 2 Make a table of values.

x	-2	-1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



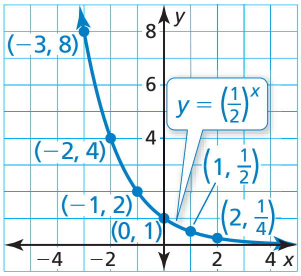
Step 3 Plot the points from the table.

Step 4 Draw, from *left to right*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the right. As $x \to -\infty$, $y \to 0$. As $x \to +\infty$, $y \to +\infty$.

b. Step 1 Identify the value of the base. The base, $\frac{1}{2}$, is greater than 0 and less than 1, so the function represents exponential decay.

Step 2 Make a table of values.

x	-3	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



Step 3 Plot the points from the table.

Step 4 Draw, from *right to left*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the left. As $x \to -\infty$, $y \to +\infty$. As $x \to +\infty$, $y \to 0$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Determine whether the function represents *exponential growth* or *exponential decay*. Then graph the function and describe the end behavior.

1. $y = 4^x$ **2.** $y = \left(\frac{2}{3}\right)^x$ **3.** $f(x) = (0.25)^x$ **4.** $f(x) = (1.5)^x$

5 MTR **5. STRUCTURE** Let $f(x) = 3^x$. What transformations of f result in exponential growth functions? What transformations result in exponential decay functions? Explain your reasoning.

Exponential Models

Some real-life quantities increase or decrease by a fixed percent each year (or some other time period). The amount y of such a quantity after t years can be modeled by one of these equations:

Exponential Growth Model $y = a(1 + r)^t$

Exponential Decay Model $y = a(1 - r)^t$

Note that a is the initial amount and r is the percent increase or decrease written as a decimal. The quantity $1 + r$ is the growth factor, and $1 - r$ is the decay factor.

GO DIGITAL

7 MTR **EXAMPLE 2** **Modeling Real Life** WATCH

The value of a car y (in thousands of dollars) can be approximated by the model $y = 25(0.85)^t$, where t is the number of years since the car was purchased.

a. Determine whether the model represents *exponential growth* or *exponential decay*.

b. Identify the annual percent increase or decrease in the value of the car.

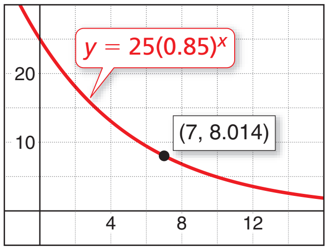
c. Estimate when the value of the car will be \$8000.

SOLUTION

a. The base, 0.85, is greater than 0 and less than 1, so the model represents exponential decay.

b. Because t is given in years and the decay factor $0.85 = 1 - 0.15$, the annual percent decrease is 0.15, or 15%.

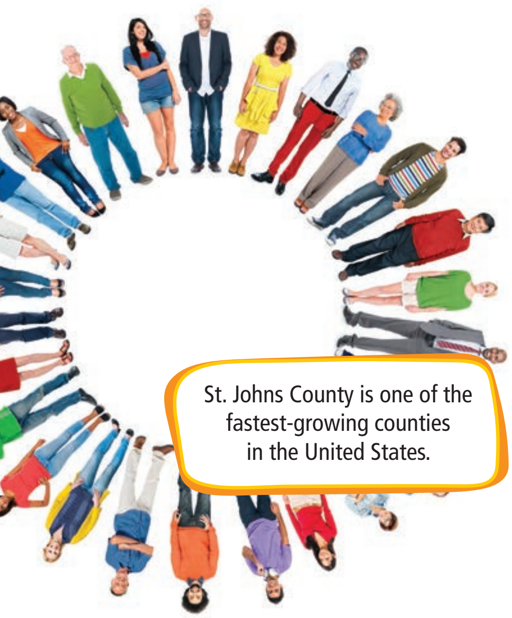
c. Use technology to determine that $y \approx 8$ when $t = 7$. After 7 years, the value of the car will be about \$8000.



7 MTR APPLY MATHEMATICS
What are the meanings of the percent decrease and the decay factor in this situation?

EXAMPLE 3 **Writing an Exponential Model** WATCH

In 2015, the population of St. Johns County, Florida, was about 226.4 thousand. During the next 4 years, the population increased by about 4% each year. Write an exponential model that represents the population y (in thousands) t years after 2015. Then estimate the population of St. Johns County in 2019.



St. Johns County is one of the fastest-growing counties in the United States.

SOLUTION

The initial amount (in thousands) is $a = 226.4$. The percent increase is $r = 0.04$, so use an exponential growth model.

$y = a(1 + r)^t$ — Write exponential growth model.

$= 226.4(1 + 0.04)^t$ — Substitute 226.4 for a and 0.04 for r.

$= 226.4(1.04)^t$ — Simplify.

To estimate the population of St. Johns County in 2019, evaluate the model when $t = 4$.

$y = 226.4(1.04)^4 \approx 264.9$ thousand.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

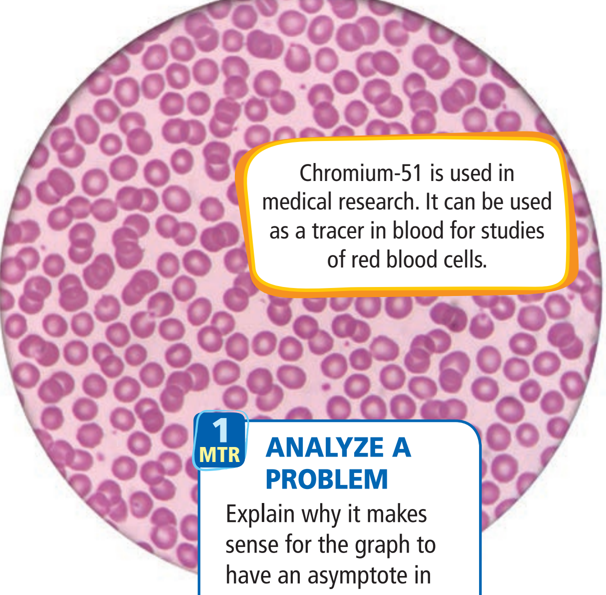
6. WHAT IF? In Example 2, the value of the car can be approximated by the model $y = 25(0.9)^t$. Identify the annual percent decrease in the value of the car. Estimate when the value of the car will be \$8000.

7. WHAT IF? In Example 3, assume that starting in 2019, the population of St. Johns County increases by 3.2% each year for the next 5 years. Use a model to estimate when the population will be 300,000.

4 MTR **8. WHICH ONE DOESN'T BELONG?** Which characteristic of an exponential decay function does *not* belong with the other three? Explain your reasoning.

base of 0.8	percent decrease of 20%	decay factor of 0.8	80% decrease

GO DIGITAL



Chromium-51 is used in medical research. It can be used as a tracer in blood for studies of red blood cells.

1 MTR ANALYZE A PROBLEM

Explain why it makes sense for the graph to have an asymptote in this situation.

EXAMPLE 4 Rewriting an Exponential Function

WATCH

The amount y (in grams) of the radioactive isotope chromium-51 remaining after t days is $y = a(0.5)^{t/28}$, where a is the initial amount (in grams). What percent of the chromium-51 decays each day?

SOLUTION

$y = a(0.5)^{t/28}$ — Write original function.

$= a[(0.5)^{1/28}]^t$ — Power of a Power Property

$\approx a(0.9755)^t$ — Evaluate power.

$= a(1 - 0.0245)^t$ — Rewrite in form $y = a(1 - r)^t$.

▶ The daily decay rate is about 0.0245, or 2.45%.

Compound interest is interest paid on an initial investment, called the *principal*, and on previously earned interest. Interest earned is often expressed as an *annual* percent, but the interest is usually compounded more than once per year. So, the exponential growth model $y = a(1 + r)^t$ must be modified for compound interest problems.

KEY IDEA

Compound Interest

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

READING

The annual rate r is often referred to as the *annual percentage rate*, or *APR*.

EXAMPLE 5 Finding the Balance in an Account

WATCH

You deposit \$9000 in an account that pays 1.46% annual interest. Find the balance after 3 years when the interest is compounded quarterly.

SOLUTION

Use the compound interest formula.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ — Write compound interest formula.

$= 9000\left(1 + \frac{0.0146}{4}\right)^{4 \cdot 3}$ — $P = 9000$, $r = 0.0146$, $n = 4$, $t = 3$

≈ 9402.21 — Use technology.

compounded quarterly

▶ The balance after 3 years is \$9402.21.

5 MTR DECOMPOSE A PROBLEM

You can write the formula as $A = 9000(1.00635)^{4t}$. Notice that this formula consists of the product of the principal, 9000, and a factor independent of the principal, $(1.00365)^{4t}$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

9. The amount y (in grams) of the radioactive isotope iodine-123 remaining after t hours is $y = a(0.5)^{t/13}$, where a is the initial amount (in grams). What percent of the iodine-123 decays each hour?

10. You deposit \$500 in an account that pays 2.5% annual interest. Find the balance after 2 years when the interest is compounded daily.

GO DIGITAL

6.1 Practice WITH CalcChat® AND CalcView®

In Exercises 1–6, evaluate the expression for (a) $x = -2$ and (b) $x = 3$.

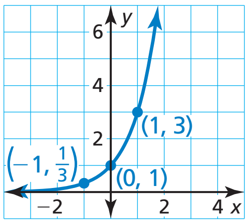
1. 2^x
2. $\left(\frac{1}{4}\right)^x$
3. $8(3)^x$
4. $6\left(\frac{1}{2}\right)^x$
5. $5(1.03)^x$
6. $-2(0.8)^x$

In Exercises 7–16, determine whether the function represents *exponential growth* or *exponential decay*. Then graph the function and describe the end behavior. *(See Example 1.)*

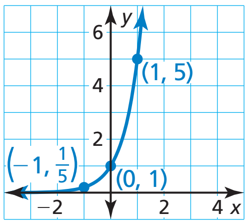
7. $y = 6^x$
8. $y = 7^x$
9. $f(x) = \left(\frac{1}{6}\right)^x$
10. $f(x) = \left(\frac{1}{8}\right)^x$
11. $y = \left(\frac{4}{3}\right)^x$
12. $y = \left(\frac{2}{5}\right)^x$
13. $f(x) = (1.2)^x$
14. $f(x) = (0.75)^x$
15. $y = (0.6)^x$
16. $y = (1.8)^x$

1 MTR **ANALYZE A PROBLEM** **In Exercises 17 and 18, use the graph of $f(x) = b^x$ to identify the value of the base b.**

17.



18.



7 MTR **19. MODELING REAL LIFE** The value of a mountain bike y (in dollars) can be approximated by the model $y = 200(0.65)^t$, where t is the number of years since the bike was purchased. *(See Example 2.)*

a. Determine whether the model represents *exponential growth* or *exponential decay*.

b. Identify the annual percent increase or decrease in the value of the bike.

c. Estimate when the value of the bike will be \$50.

7 MTR **20. MODELING REAL LIFE** The population P (in millions) of Peru during a recent decade can be approximated by $P = 28.22(1.01)^t$, where t is the number of years since the beginning of the decade.

a. Determine whether the model represents *exponential growth* or *exponential decay*.

b. Identify the annual percent increase or decrease in population.

c. Estimate when the population was about 30 million.

7 MTR **21. MODELING REAL LIFE** In 2014, there were about 7 billion cell phone subscribers in the world. During the next 5 years, the number of cell phone subscribers increased by about 3% each year. *(See Example 3.)*

a. Write an exponential model that represents the number of cell phone subscribers y (in billions) t years after 2014. Then estimate the number of cell phone subscribers in 2019.

b. Estimate when the number of cell phone subscribers was about 7.5 billion.

7 MTR **22. MODELING REAL LIFE** You take a 325-milligram dosage of ibuprofen. The amount of medication in your bloodstream decreases by about 29% each hour.

a. Write an exponential model that represents the amount y (in milligrams) of ibuprofen in your bloodstream t hours after the initial dose.

b. Estimate when you will have 100 milligrams of ibuprofen in your bloodstream.

6 MTR **JUSTIFYING STEPS** **In Exercises 23 and 24, justify each step in rewriting the exponential function.**

23. $y = a(3)^{t/14}$ — Write original function.

$= a[(3)^{1/14}]^t$

$\approx a(1.0816)^t$

$= a(1 + 0.0816)^t$

24. $y = a(0.1)^{t/3}$ — Write original function.

$= a[(0.1)^{1/3}]^t$

$\approx a(0.4642)^t$

$= a(1 - 0.5358)^t$

GO DIGITAL

25. **MODELING REAL LIFE** When a plant or an animal dies, it stops acquiring carbon-14 from the atmosphere. The amount y (in grams) of carbon-14 in the body of an organism t years after the organism dies is $y = a(0.5)^{t/5730}$, where a is the initial amount (in grams). What percent of the carbon-14 is released each year? *(See Example 4.)*

26. **MODELING REAL LIFE** The number y of *Salmonella* cells on an egg after t minutes is $y = a(4.7)^{t/45}$, where a is the initial number of cells. By what percent does the number of *Salmonella* cells increase each minute?

In Exercises 27–34, rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. State the growth or decay rate, and describe the end behavior of the function.

27. $y = a(2)^{t/3}$

28. $y = a(4)^{t/6}$

29. $y = a(0.5)^{t/12}$

30. $y = a(0.25)^{t/9}$

31. $y = a\left(\frac{2}{3}\right)^{t/10}$

32. $y = a\left(\frac{5}{4}\right)^{t/22}$

33. $y = a(2)^{8t}$

34. $y = a\left(\frac{1}{3}\right)^{3t}$

35. **PROBLEM SOLVING** You deposit \$5000 in an account that pays 2.25% annual interest. Find the balance after 5 years when the interest is compounded quarterly. *(See Example 5.)*

36. **REASONING** You deposit \$2200 into each of three separate bank accounts that pay 3% annual interest. Interest is compounded quarterly in Account 1, monthly in Account 2, and daily in Account 3. How much interest does each account earn after 6 years?

37. **ERROR ANALYSIS** You invest \$500 in the stock of a company. The value of the stock decreases 2% each year. Describe and correct the error in writing a model for the value of the stock after t years.

$$y = \left(\begin{matrix}\text{Initial}\\ \text{amount}\end{matrix}\right)\left(\begin{matrix}\text{Decay}\\ \text{factor}\end{matrix}\right)^t$$

$$y = 500(0.02)^t$$

38. **ERROR ANALYSIS** You deposit \$250 in an account that pays 1.25% annual interest. Describe and correct the error in finding the balance after 3 years when the interest is compounded quarterly.

$$A = 250\left(1 + \frac{1.25}{4}\right)^{4 \cdot 3}$$

$$= \$6533.29$$

GO DIGITAL

In Exercises 39–42, use the given information to find the balance in the account earning compound interest after 6 years when the principal is \$3500.

39. $r = 2.16\%$, compounded quarterly

40. $r = 2.29\%$, compounded monthly

41. $r = 1.26\%$, compounded monthly

42. $r = 1.83\%$, compounded daily

43. **STRUCTURE** In the compound interest formula for interest compounded yearly, $A = P(1 + r)^t$, what does P represent? What does $(1 + r)^t$ represent? Does P depend on $(1 + r)^t$? Does $(1 + r)^t$ depend on P? Explain.

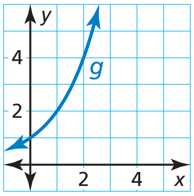
44. **STRUCTURE** The equation $y = 1000(1.12)^t$ represents the balance y (in dollars) in an account after t years.

a. What is the annual interest rate? Explain.

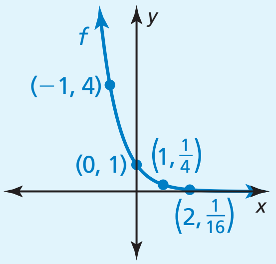
b. Rewrite the function in the form $y = 1000b^{12t}$.

c. Explain how you can use your equation in part (b) to find the monthly interest rate.

45. **COMPARING FUNCTIONS** Compare the graph of $f(x) = 2(1.5)^x$ to the graph of g. Which graph has a greater y-intercept? Which graph is increasing at a faster rate? Explain your reasoning.



46. **HOW DO YOU SEE IT?** The graph shows an exponential function of the form $f(x) = ab^x$.



a. Determine whether f represents *exponential growth* or *exponential decay*.

b. What are the domain and range of the function? Explain.

47. **PROBLEM SOLVING** The population p of a small town after x years can be modeled by the function $p = 6850(1.03)^x$. What is the average rate of change in the population over the first 6 years? Justify your answer.

48. **THOUGHT PROVOKING** The function $f(x) = b^x$ represents exponential decay. Write another exponential decay function in terms of b and x.

49. **COLLEGE PREP** The radioactive isotope phosphorus-32 has a daily decay rate of 4.8%. After 1 week, there are 7.1 grams of phosphorus-32 remaining. About how many grams were there initially?

Ⓐ 1.2 g
Ⓑ 5.1 g
Ⓒ 10.0 g
Ⓓ 1209.4 g

50. **REASONING** Consider the exponential function $f(x) = ab^x$.

a. Show that $\dfrac{f(x+1)}{f(x)} = b$.

b. Use the equation in part (a) to explain why there is no exponential function of the form $f(x) = ab^x$ whose graph passes through the points in the table.

x	0	1	2	3	4
y	4	4	8	24	72

51. **PROBLEM SOLVING** The number E of eggs a Leghorn chicken produces per year can be modeled by the equation $E = 179.2(0.89)^{w/52}$, where w is the age (in weeks) of the chicken and $w \geq 22$.

a. Identify the decay factor and the percent decrease.

b. Graph the equation.

c. Estimate the egg production of a chicken that is 2.5 years old.

d. Explain how you can rewrite the given equation so that time is measured in years rather than in weeks.

52. **DIG DEEPER** You buy a new laptop for $1300 and sell it 4 years later for $275. Assume that the resale value of the laptop decays exponentially with time. Write an equation that represents the resale value V (in dollars) of the laptop as a function of the time t (in years) since it was purchased.

REVIEW & REFRESH

WATCH

In Exercises 53–56, simplify the expression.

53. $x^9 \cdot x^2$

54. $(2x \cdot 3x^5)^3$

55. $\left(\dfrac{4x^8}{2x^6}\right)^4$

56. $\dfrac{12x}{4x} + 5x$

5 MTR **57.** **STRUCTURE** In the exponential model $y = 2.4(1.5)^x$, identify the initial amount, the growth or decay factor, and the percent increase or decrease.

In Exercises 58 and 59, determine whether the functions are inverse functions.

58. $f(x) = -\dfrac{1}{2}x + 5,\ g(x) = -2x + 10$

59. $f(x) = \sqrt[3]{4x},\ g(x) = \dfrac{x^3}{64}$

60. Let $f(x) = -x^3$ and $g(x) = x + 4$. Find $(fg)(x)$ and $\left(\dfrac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(4)$ and $\left(\dfrac{f}{g}\right)(4)$.

In Exercises 61 and 62, determine whether the function represents *exponential growth* or *exponential decay*. Then graph the function and describe the end behavior.

61. $y = (3.25)^x$

62. $y = \left(\dfrac{3}{4}\right)^x$

In Exercises 63 and 64, use the functions $f(x) = x^2 - 1$ and $g(x) = 2x - 3$ to complete the table.

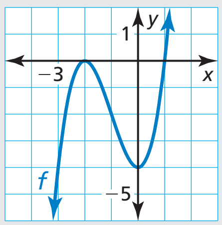
63.

x	$f(g(x))$
0	
1	
2	
3	
4	

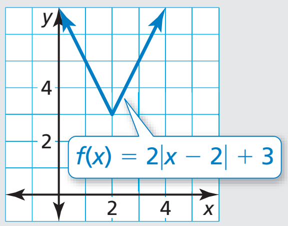
64.

x	$g(f(x))$
−2	
−1	
0	
1	
2	

65. Describe the x-values for which (a) f is increasing, (b) f is decreasing, (c) $f(x) > 0$, and (d) $f(x) < 0$.



66. Identify the function family to which f belongs. Compare the graph of f to the graph of its parent function.



GO DIGITAL