5.4 Graphing Cube Root Functions

Learning Target: Graph and describe cube root functions.

Success Criteria:
- I can graph cube root functions and describe their characteristics.
- I can graph and describe transformations of cube root functions.
- I can use cube root functions to solve real-life problems.

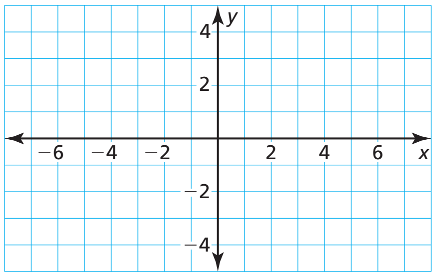
EXPLORE IT! Graphing Cube Root Functions

Work with a partner.

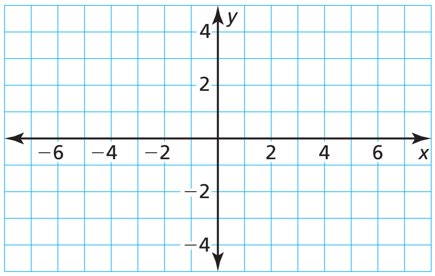
a. Graph $f(x) = \sqrt[3]{x}$. Find the domain and range of the function. Then make several observations about the graph.

b. Describe what you expect the graph of g to look like. Then sketch the graph of g and find the domain and range of the function.

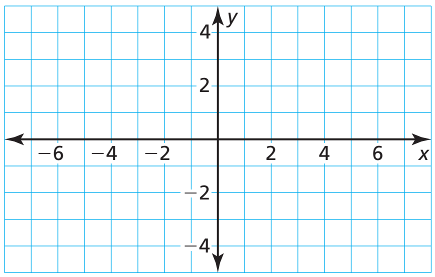
i. $g(x) = 2\sqrt[3]{x}$



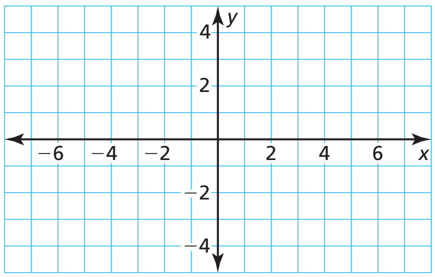
ii. $g(x) = \sqrt[3]{x} - 3$



iii. $g(x) = -\sqrt[3]{x}$



iv. $g(x) = \sqrt[3]{x + 3}$



c. Without graphing, compare the graph of $g(x) = -\sqrt[3]{x + 3} + 1$ to the graph of $f(x) = \sqrt[3]{x}$. Explain your reasoning.

d. How are graphs of cube root functions similar to graphs of other types of functions? How are they different?

4 MTR CONSTRUCT AN ARGUMENT

All real numbers have a cube root. Does this imply that the domain of each function in part (b) is all real numbers? Explain.

Algebraic Reasoning

MA.912.AR.7.2 Given a table, equation or written description of a square root or cube root function, graph that function and determine its key features.

Functions

MA.912.F.2.2 Identify the effect on the graph of a given function of two or more transformations defined by adding a real number to the x- or y-values or multiplying the x- or y-values by a real number.

Also MA.912.AR.7.3, MA.912.F.2.3, MA.912.F.2.5

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Describing and Graphing Cube Root Functions

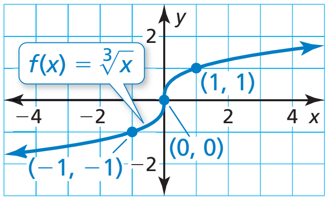
Vocabulary

cube root function, *p. 266*

KEY IDEA

Cube Root Functions

A **cube root function** is a radical function with an index of 3. The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$. The domain and range of f are all real numbers.



EXAMPLE 1 **Describing Characteristics**

Graph $f(x) = \sqrt[3]{x} + 1$. Determine when the function is positive, negative, increasing, or decreasing. Then describe the end behavior of the function.

SOLUTION

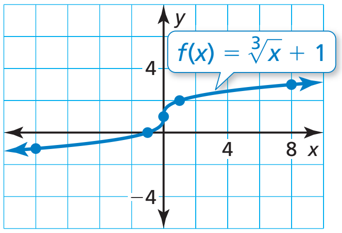
Step 1 Make a table of values.

x	−8	−1	0	1	8
$f(x)$	−1	0	1	2	3

5 MTR USE STRUCTURE Explain how to choose convenient x-values when making the table.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



Positive and Negative: The x-intercept is −1. The function is negative over the interval $(-\infty, -1)$ and positive over the interval $(-1, \infty)$.

Increasing and Decreasing: The domain is all real numbers, and the function increases over its entire domain.

End Behavior: From the graph, you can see that $y \to -\infty$ as $x \to -\infty$ and $y \to +\infty$ as $x \to +\infty$.

SELF-ASSESSMENT 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function. Determine when the function is positive, negative, increasing, or decreasing. Then describe the end behavior of the function.

1. $h(x) = \sqrt[3]{x} + 3$
2. $m(x) = \sqrt[3]{x - 5}$
3. $g(x) = 4\sqrt[3]{x}$

4 MTR
4. **WHICH ONE DOESN'T BELONG?** Which function does *not* belong with the other three? Explain your reasoning.

$g(x) = 4\sqrt[3]{x} + 1$ | $h(x) = \sqrt[3]{x + 2}$ | $m(x) = \sqrt[3]{x - 3}$ | $n(x) = 3\sqrt{x} - 5$

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You can transform graphs of cube root functions in the same way you transformed graphs of square root functions.

EXAMPLE 2 Comparing Graphs of Cube Root Functions

WATCH

Graph $g(x) = -\sqrt[3]{x+2}$. Compare the graph to the graph of $f(x) = \sqrt[3]{x}$.

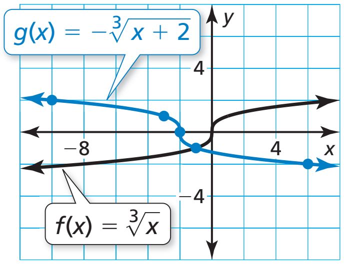
SOLUTION

Step 1 Make a table of values.

x	-10	-3	-2	-1	6
$g(x)$	2	1	0	-1	-2

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



The graph of g is a translation 2 units left and a reflection in the x-axis of the graph of f.

EXAMPLE 3 Graphing $y = a\sqrt[3]{x-h} + k$

WATCH

Graph $g(x) = 2\sqrt[3]{x-3} + 4$. Describe the transformations from the graph of $f(x) = \sqrt[3]{x}$ to the graph of g.

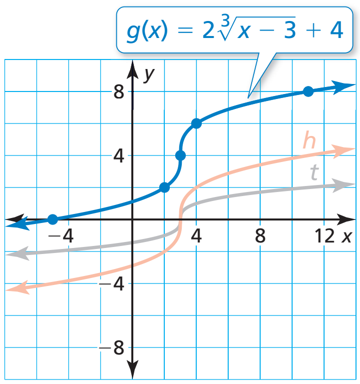
SOLUTION

Notice that you can rewrite g as $g(x) = 2f(x-3) + 4$.

Step 1 Translate the graph of f horizontally 3 units right to get the graph of $t(x) = \sqrt[3]{x-3}$.

Step 2 Stretch the graph of t vertically by a factor of 2 to get the graph of $h(x) = 2\sqrt[3]{x-3}$.

Step 3 Translate the graph of h vertically 4 units up to get the graph of $g(x) = 2\sqrt[3]{x-3} + 4$.



The graph of g is a translation 3 units right, a vertical stretch by a factor of 2, and a translation 4 units up of the graph of f.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function. Compare the graph to the graph of $f(x) = \sqrt[3]{x}$.

5. $g(x) = \sqrt[3]{0.5x} + 5$

6. $h(x) = 4\sqrt[3]{x-1}$

7. $n(x) = \sqrt[3]{4-x}$

8. Graph $g(x) = -\frac{1}{2}\sqrt[3]{x+2} - 4$. Describe the transformations from the graph of $f(x) = \sqrt[3]{x}$ to the graph of g.

9. **WRITING** Explain why the domain and range of $y = a\sqrt[3]{x-h} + k$ are all real numbers.

10. **REASONING** Compare the graphs of $y = \sqrt[3]{-x}$ and $y = -\sqrt[3]{x}$. What does this mean in terms of transformations?

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EXAMPLE 4 Describing a Transformation

The table approximates two cube root functions f and g. Describe the transformation from the graph of f to the graph of g.

x	-2	-1	0	1
$f(x)$	-2.52	-2	0	2
$g(x)$	-2.88	-2.52	-2	0

SOLUTION

To determine the transformation, compare values of $f(x)$ and $g(x)$.

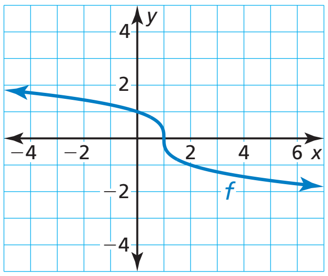
x	-2	-1	0	1
$f(x)$	-2.52	-2	0	2

$g(x)$	-2.88	-2.52	-2	0

For each value of x, $g(x)$ is equal to the value of $f(x - 1)$. So, $g(x) = f(x - 1)$.

▶ The graph of g is a translation 1 unit right of the graph of f.

EXAMPLE 5 Representing a Transformation



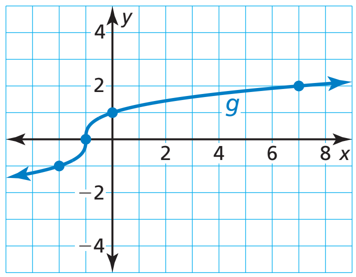
The graph of a cube root function f is shown. The graph of g is a reflection in the y-axis of the graph of f. Graph g.

SOLUTION

Because the graph of g is a reflection in the y-axis of the graph of f, $g(x) = f(-x)$. So, the graph of g consists of the points $(x, f(-x))$.

Step 1 Use the graph of f to make a table of values for the ordered pairs $(x, f(-x))$.

x	$-x$	$f(-x)$
-2	2	-1
-1	1	0
0	0	1
7	-7	2



Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

4 MTR CONSTRUCT AN ARGUMENT

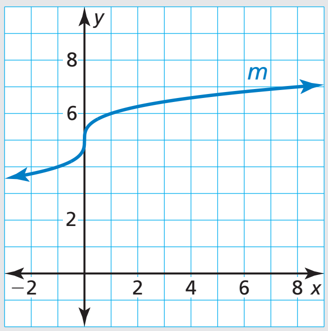
Explain why you could also use the points $(-x, f(x))$ to graph g.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

11. The table approximates two cube root functions f and g. Describe the transformation from the graph of f to the graph of g.

x	-1	0	1	2
$f(x)$	-2	1	4	4.78
$g(x)$	2	-1	-4	-4.78



12. The graph of a cube root function m is shown. The graph of n is a translation 4 units down of the graph of m. Graph n.

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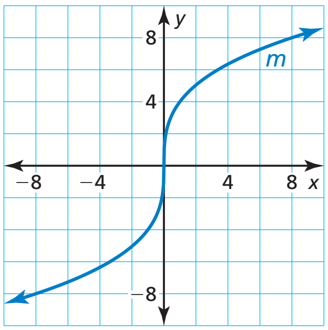
Comparing Average Rates of Change

EXAMPLE 6 Comparing Cube Root Functions

WATCH



The graph of cube root function m is shown. Compare the average rate of change of m to the average rate of change of $h(x) = \sqrt[3]{\frac{1}{4}x}$ over the interval $[0, 8]$.

SOLUTION

To calculate the average rates of change, use points whose x-coordinates are 0 and 8.

Function m: Use the graph to estimate. Use $(0, 0)$ and $(8, 8)$.

$$\frac{m(8) - m(0)}{8 - 0} \approx \frac{8 - 0}{8} = 1 \qquad \text{Average rate of change of } m$$

Function h: Evaluate h when $x = 0$ and $x = 8$.

$$h(0) = \sqrt[3]{\frac{1}{4}(0)} = 0 \qquad \text{and} \qquad h(8) = \sqrt[3]{\frac{1}{4}(8)} = \sqrt[3]{2}$$

Use $(0, 0)$ and $\left(8, \sqrt[3]{2}\right)$.

$$\frac{h(8) - h(0)}{8 - 0} = \frac{\sqrt[3]{2} - 0}{8} \approx 0.16 \qquad \text{Average rate of change of } h$$

▶ Because $1 > 0.16$, the average rate of change of m is greater than the average rate of change of h over the interval $[0, 8]$.

Solving Real-Life Problems

EXAMPLE 7 Modeling Real Life

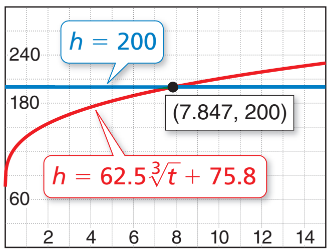
7 MTR WATCH

The shoulder height h (in centimeters) of a male Asian elephant can be modeled by the function $h = 62.5\sqrt[3]{t} + 75.8$, where t is the age (in years) of the elephant. Estimate the age of an elephant whose shoulder height is 200 centimeters.

SOLUTION

The solution is the t-value for which $h = 200$. So, solve $200 = 62.5\sqrt[3]{t} + 75.8$.

One way to find the solution is to graph each side of the equation, as shown. Use technology to find the coordinates of the point of intersection.



The two graphs intersect at about $(8, 200)$.

▶ So, the elephant is about 8 years old.

SELF-ASSESSMENT 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

13. In Example 6, compare the average rate of change of m to the average rate of change of $g(x) = 8\sqrt[3]{x}$ over the interval $[2, 8]$.

14. WHAT IF? Estimate the age of an elephant whose shoulder height is 175 centimeters.

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5.4 Practice WITH CalcChat® AND CalcView®

In Exercises 1–6, graph the function. Determine when the function is positive, negative, increasing or decreasing. Then describe the end behavior of the function. *(See Example 1.)*

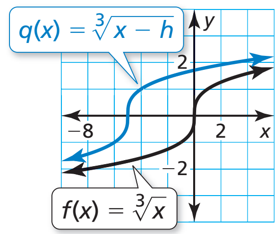
▶ **1.** $h(x) = \sqrt[3]{x-4}$ **2.** $g(x) = \sqrt[3]{x+1}$

3. $m(x) = \sqrt[3]{x} + 5$ **4.** $q(x) = \sqrt[3]{x} - 3$

5. $p(x) = 6\sqrt[3]{x}$ **6.** $j(x) = \sqrt[3]{\frac{1}{2}x}$

In Exercises 7–10, compare the graphs. Find the value of *h*, *k*, or *a*.

7.



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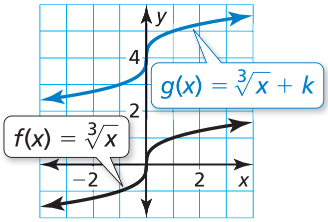

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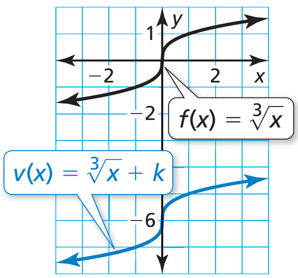

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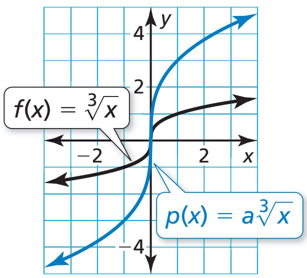


In Exercises 11–20, graph the function. Compare the graph to the graph of $f(x) = \sqrt[3]{x}$. *(See Example 2.)*

11. $r(x) = -\sqrt[3]{x-2}$ **12.** $h(x) = -\sqrt[3]{x} + 3$

13. $k(x) = 5\sqrt[3]{x+1}$ **14.** $j(x) = 0.5\sqrt[3]{x-4}$

▶ **15.** $g(x) = 4\sqrt[3]{x} - 3$ **16.** $m(x) = 3\sqrt[3]{x} + 7$

17. $n(x) = \sqrt[3]{-8x} - 1$ **18.** $v(x) = \sqrt[3]{5x} + 2$

19. $q(x) = \sqrt[3]{2(x+3)}$ **20.** $p(x) = \sqrt[3]{3(1-x)}$

In Exercises 21–26, graph the function. Describe the transformations from the graph of $f(x) = \sqrt[3]{x}$ to the graph of the given function. *(See Example 3.)*

21. $g(x) = \sqrt[3]{x-4} + 2$ **22.** $n(x) = \sqrt[3]{x+1} - 3$

▶ **23.** $j(x) = -5\sqrt[3]{x+3} + 2$ **24.** $k(x) = 6\sqrt[3]{x-9} - 5$

25. $v(x) = \frac{1}{3}\sqrt[3]{x-1} + 7$ **26.** $h(x) = -\frac{3}{2}\sqrt[3]{x+4} - 3$

In Exercises 27–30, write a rule for *g* described by the transformations of the graph of *f*.

27. The graph of g is a translation 3 units left of the graph of $f(x) = \sqrt[3]{x}$.

28. The graph of g is a vertical stretch by a factor of 2 of the graph of $f(x) = \sqrt[3]{x+3}$.

29. Let g be a horizontal shrink by a factor of 0.5, followed by a translation 4 units down of the graph of $f(x) = \sqrt[3]{2x+6}$.

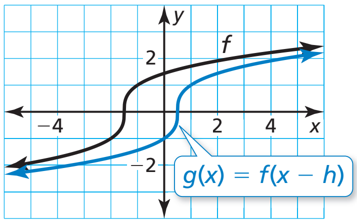
30. Let g be a reflection in the y-axis, followed by a translation 1 unit right of the graph of $f(x) = 2\sqrt[3]{x-1}$.

In Exercises 31–36, describe the transformation from the graph of *f* to the graph of *g*. *(See Example 4.)*

▶ **31.**

x	-8	-1	0	1
$f(x)$	0	1	2	3
$g(x) = f(x) + k$	-4	-3	-2	-1

32.



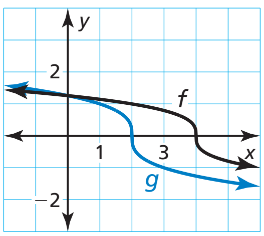
33.

x	-5	-3	-1	1
$f(x)$	-1	1	1.44	1.71
$g(x)$	1	-1	-1.44	-1.71

34.

x	-2	0	2	4
$f(x)$	-2	0	2	2.52
$g(x)$	6	0	-6	-7.56

35.



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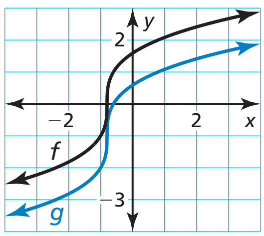

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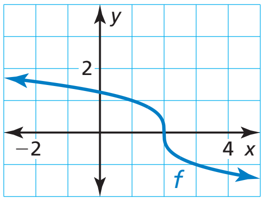
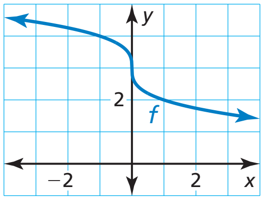
In Exercises 37–40, the graph of a cube root function *f* is shown. Graph *g*. *(See Example 5.)*

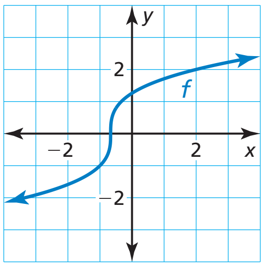
37. The graph of g is a translation 3 units up of the graph of f.



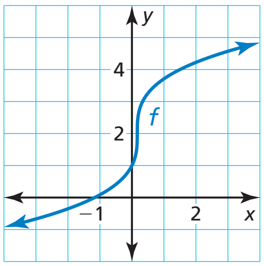
38. The graph of g is a translation 2 units left of the graph of f.



39. The graph of g is a vertical stretch by a factor of 4 of the graph of f.



40. The graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ of the graph of f.



In Exercises 41 and 42, the table of values represents a cube root function. Create a table of values for *g*.

41. The graph of g is a reflection in the x-axis of the graph of f.

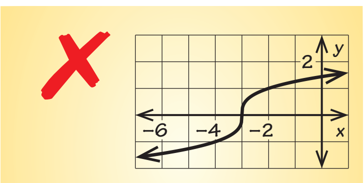
x	−7	0	1	2	9
$f(x)$	0	1	2	3	4

42. The graph of g is a translation 6 units right of the graph of f.

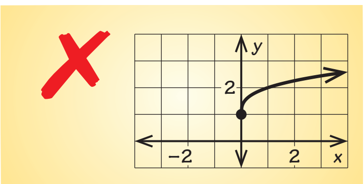
x	−13.5	−4	0	4	13.5
$f(x)$	0	−1	−3	−5	−6

4 MTR **ERROR ANALYSIS In Exercises 43 and 44, describe and correct the error in graphing the function.**

43. $f(x) = \sqrt[3]{x - 3}$



44. $h(x) = \sqrt[3]{x} + 1$



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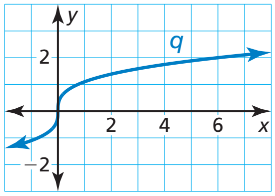
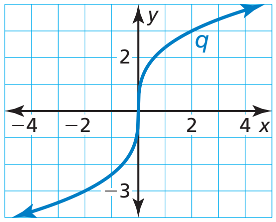
In Exercises 45 and 46, compare the average rate of change of *f* to the average rate of change of *q* over the given interval. *(See Example 6.)*

45. $f(x) = 3\sqrt[3]{x}$; $[0, 6]$



46. $f(x) = \sqrt[3]{0.5x}$; $[-2, 2]$

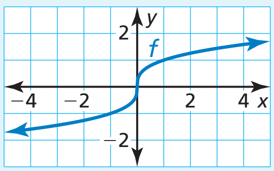


7 MTR **47. MODELING REAL LIFE** The radius r of a sphere is given by $r = \sqrt[3]{\frac{3}{4\pi}V}$, where V is the volume of the sphere. Estimate the volume of a spherical head of brain coral with a radius of 1.5 feet. *(See Example 7.)*

7 MTR **48. MODELING REAL LIFE** For a drag race car that weighs 1600 kilograms, the velocity v (in kilometers per hour) reached by the end of a drag race can be modeled by the function $v = 23.8\sqrt[3]{p}$, where p is the car's power (in horsepower). Estimate the power of a 1600-kilogram car that reaches a velocity of 220 kilometers per hour.

4 MTR **49. MAKING AN ARGUMENT** Your friend says that all cube root functions are odd functions. Is your friend correct? Explain.

50. HOW DO YOU SEE IT? The graph represents the cube root function $f(x) = \sqrt[3]{x}$.



a. On what interval is f negative? positive?

b. On what interval, if any, is f decreasing? increasing?

c. Does f have a maximum or minimum value? Explain.

d. Describe the end behavior of f.

e. Find the average rate of change of f over the interval $[-1, 1]$.

51. **REASONING** Can a cube root function be increasing on part of its domain and decreasing on a different part of its domain? Explain your reasoning.

52. **COLLEGE PREP** Which of the following describes the transformations from the graph of $f(x) = \sqrt[3]{x+5} + 2$ to the graph of $g(x) = \sqrt[3]{x-1} - 2$?

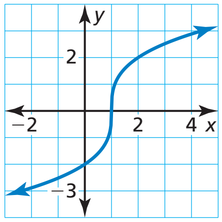
(A) a translation 1 unit right and 2 units down

(B) a translation 1 unit left and 2 units down

(C) a translation 6 units right and 4 units down

(D) a translation 6 units left and 4 units down

53. **REASONING** Write the cube root function represented by the graph.



54. **THOUGHT PROVOKING** Write a cube root function that passes through the point (3, 4) and has an average rate of change of −1 over the interval [−5, 2].

REVIEW & REFRESH

WATCH

In Exercises 55–58, factor the polynomial.

55. $x^2 - 20x + 75$

56. $3x^2 + 12x - 36$

57. $2x^2 - 11x + 9$

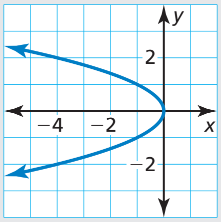
58. $4x^2 + 7x - 15$

In Exercises 59 and 60, determine when the function is positive, negative, increasing, or decreasing. Then describe the end behavior of the function.

59. $f(x) = 15\sqrt{x}$

60. $y = -\sqrt{3-x} + 2$

61. Determine whether the graph represents a function. Explain.



7 MTR 62. **MODELING REAL LIFE** You can use at most 5 gigabytes of data per month on your cell phone. Your data usage so far for the month is 1.24 gigabytes. What are the possible amounts of data you can use for the remainder of the month?

In Exercises 63–66, solve the equation.

63. $x^2 - 36 = 0$

64. $5x^2 + 20 = 0$

65. $(x + 4)^2 = 81$

66. $25(x - 2)^2 = 9$

In Exercises 67 and 68, graph the function. Compare the graph to the graph of $f(x) = \sqrt[3]{x}$.

67. $g(x) = \sqrt[3]{x+7}$

68. $h(x) = -\sqrt[3]{x} - 6$

In Exercises 69 and 70, solve the inequality.

69. $x^2 + 7x + 12 < 0$

70. $x^2 - 10x + 25 \geq 4$

In Exercises 71 and 72, graph the function. Label the vertex and axis of symmetry.

71. $g(x) = -(x + 4)^2 - 1$

72. $h(x) = 4x^2 + 8x - 5$

In Exercises 73 and 74, write a function g whose graph represents the indicated transformation of the graph of f.

73. $f(x) = x$; vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the y-axis

74. $f(x) = |x + 1| - 3$; horizontal stretch by a factor of 9.

In Exercises 75–77, factor the polynomial completely.

75. $w^3 - 6w^2 - 9w + 54$

76. $4c^4 - 16c^3 + 16c^2$

77. $54x^5 - 6x$

In Exercises 78 and 79, evaluate the expression using technology. Round your answer to two decimal places, if necessary.

78. $\sqrt[5]{12{,}392}$

79. $95^{-2/3}$

In Exercises 80 and 81, simplify the expression.

80. $\sqrt[3]{64p^3y^{12}}$

81. $\sqrt[5]{\frac{t^{20}}{u^{10}}}$

GO DIGITAL