# 4.8 Modeling with Polynomial Functions

**Learning Target:** Write polynomial functions.

**Success Criteria:**
- I can write a polynomial function given a graph or a set of points.
- I can write a polynomial function using finite differences.
- I can use technology to find a polynomial model for a set of data.

## EXPLORE IT! Modeling Real-Life Data

**Work with a partner.** The data show the prices per share *y* (in dollars) for Amazon.com, Inc., stock *t* years after 2000.

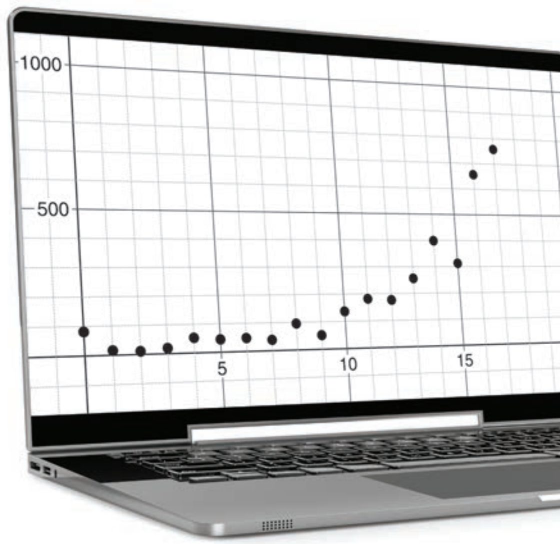
| *t* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| *y* | 81.50 | 15.81 | 10.93 | 19.19 | 52.76 | 44.95 | 47.47 | 38.68 |

| *t* | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|
| *y* | 95.35 | 51.35 | 136.25 | 181.37 | 175.89 | 256.08 | 398.80 | 312.58 |

| *t* | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|
| *y* | 656.29 | 757.92 | 1172.00 | 1465.20 | 1875.00 |

**a.** Use technology to make a scatter plot of the data. Describe the scatter plot.

**b.** Use technology to find a linear model and a quadratic model to represent the data. Is either model a good fit? How can you tell?




**c.** Is there another type of model you can use that better represents the data in the table? Use technology to find the model, and explain why it is a better fit. Compare your results with your classmate's.

**d.** Can you use the model you found in part (c) to make predictions about the share prices for Amazon.com, Inc., for future years? Explain your reasoning.

**e.** How can you tell when a model fits a set of data *exactly*?

**2 MTR MODEL A PROBLEM**
How can you determine whether one model is a better fit for a set of data than another model?

**Functions**
**MA.912.F.1.1** Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it.

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## Writing a Polynomial Function for a Set of Points

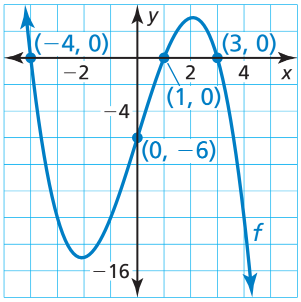
**Vocabulary**

finite differences, *p. 224*

You know that two points determine a line and three points not on a line determine a parabola. In Example 1, you will see that four points not on a line or a parabola determine the graph of a cubic function.

**EXAMPLE 1** **Writing a Cubic Function**

Write the cubic function whose graph is shown.




**SOLUTION**

**Step 1** Use the three $x$-intercepts to write the function in intercept form.

$$f(x) = a(x + 4)(x - 1)(x - 3)$$

**Step 2** Find the value of $a$ by substituting the coordinates of the point $(0, -6)$.

$$-6 = a(0 + 4)(0 - 1)(0 - 3)$$

$$-6 = 12a$$

$$-\frac{1}{2} = a$$

▶ The function is $f(x) = -\frac{1}{2}(x + 4)(x - 1)(x - 3)$.

**REMEMBER**

The intercept form of a cubic function is

$$f(x) = a(x - p)(x - q)(x - r)$$

where $a \neq 0$, and the $x$-intercepts of the graph of $f$ are $p$, $q$, and $r$.

**Check**

Check the end behavior of $f$. The degree of $f$ is odd and $a < 0$. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$, which matches the graph. ✓

**SELF-ASSESSMENT** 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

**Write a cubic function whose graph passes through the given points.**

1. $(-4, 0), (0, 10), (2, 0), (5, 0)$
2. $(-1, 0), (0, -12), (2, 0), (3, 0)$

## Finite Differences

When the $x$-values in a data set are equally spaced, the differences of consecutive $y$-values are called **finite differences**. Recall from Section 2.4 that the first and second differences of $y = x^2$ are as follows:

Equally spaced $x$-values

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

first differences: −5 −3 −1 1 3 5

second differences: 2 2 2 2 2

Notice that $y = x^2$ has degree *two* and that the *second* differences are constant and nonzero. This illustrates the first of the two properties of finite differences shown on the next page.

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## KEY IDEA

### Properties of Finite Differences

1. If a polynomial function $y = f(x)$ has degree $n$, then the $n$th differences of function values for equally spaced $x$-values are nonzero and constant.
2. Conversely, if the $n$th differences of equally spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree $n$.

The second property of finite differences allows you to write a polynomial function that models a set of equally spaced data.

### EXAMPLE 2 Writing a Function Using Finite Differences

WATCH

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −3 | −9 | −14 | 0 | 63 | 217 | 516 |

### SOLUTION

**Step 1** Write the function values. Find the first differences by subtracting consecutive values. Then find the second differences by subtracting consecutive first differences. Continue until you obtain differences that are nonzero and constant.

$f(1)$ $f(2)$ $f(3)$ $f(4)$ $f(5)$ $f(6)$ $f(7)$

−3 −9 −14 0 63 217 516 — Write function values for equally spaced $x$-values.

−6 −5 14 63 154 299 — First differences

1 19 49 91 145 — Second differences

18 30 42 54 — Third differences

12 12 12 — Fourth differences

Because the fourth differences are nonzero and constant, you can model the data *exactly* with a quartic function.

**Step 2** Use technology to enter the data from the table. Then use *quartic regression* to obtain a polynomial function.

$y = ax^4 + bx^3 + cx^2 + dx + f$

PARAMETERS

$a = 0.5$ $b = -2$

$c = 0$ $d = 0.5$

$f = -2$

STATISTICS

$R^2 = 1$

A quartic function that fits the data exactly is

$$f(x) = \frac{1}{2}x^4 - 2x^3 + \frac{1}{2}x - 2.$$

**5 MTR USE STRUCTURE**

Factor $f(x)$ completely. Explain your procedure to a classmate.

## SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

**3. WRITING** Explain how you know when a set of data can be modeled by a polynomial function of degree $n$.

**4.** Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| $f(x)$ | 6 | 15 | 22 | 21 | 6 | −29 |

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## Finding Models Using Technology

In Examples 1 and 2, you found a model that *exactly* fits a set of data. In many real-life situations, you cannot find models to fit data exactly. Despite this limitation, you can still use technology to approximate the data with a polynomial model.

7 MTR

### EXAMPLE 3 Modeling Real Life

WATCH

The data show the numbers $y$ of active bald eagle nests in Florida $t$ years after 2000. Find a model for the data. Use the model to estimate the number of active bald eagle nests in 2018.

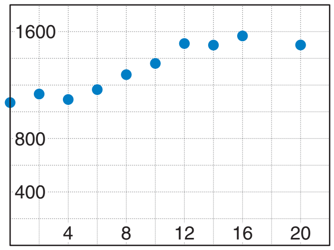
| $t$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $y$ | 1069 | 1133 | 1092 | 1166 | 1278 |

| $t$ | 10 | 12 | 14 | 16 | 20 |
|---|---|---|---|---|---|
| $y$ | 1362 | 1511 | 1499 | 1568 | 1500 |

Florida has one of the densest concentrations of bald eagles in the United States. The Florida Fish and Wildlife Conservation Commission removed the bald eagle from its imperiled species list in 2008.

**SOLUTION**

**Step 1** Use technology to make a scatter plot of the data. The data suggest some type of polynomial model such as a cubic function.




**Step 2** Use *cubic regression*. The coefficients can be rounded to obtain the following:

$$y = -0.268t^3 + 7.26t^2 - 17.6t + 1089 \qquad \text{Cubic model}$$

$y = ax^3 + bx^2 + cx + d$

PARAMETERS

$a = -0.268028 \quad b = 7.26447$

$c = -17.6201 \quad d = 1088.86$

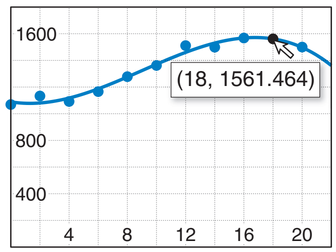
STATISTICS

$R^2 = 0.9768$

**Step 3** Graph the equation with the data. The graph shows that the model is a good fit for the data.

**Step 4** Find $y$ when $t = 18$. It is about 1561.

▶ You can estimate that the number of active bald eagle nests in 2018 was about 1561.




## SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

**5.** Use technology to find a polynomial function that fits the data.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 5 | 13 | 17 | 11 | 11 | 56 |

**6.** Use the model in Example 3 to estimate the number of active bald eagle nests in 2013.

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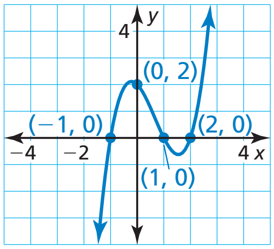
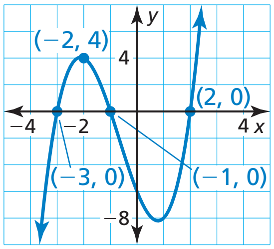
# 4.8 Practice WITH CalcChat® AND CalcView®

**In Exercises 1–4, write a cubic function whose graph passes through the given points.** *(See Example 1.)*

**1.**




**2.**




**3.** (−5, 0), (1, 0), (2, −2), (4, 0)

**4.** (−6, 0), (−3, 0), (0, −9), (3, 0)

**In Exercises 5–10, use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.** *(See Example 2.)*

**5.**

| $x$ | −6 | −3 | 0 | 3 | 6 | 9 |
|---|---|---|---|---|---|---|
| $f(x)$ | −2 | 15 | −4 | 49 | 282 | 803 |

**6.**

| $x$ | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $f(x)$ | −14 | −5 | −2 | 7 | 34 | 91 |

**7.** (−4, −317), (−3, −37), (−2, 21), (−1, 7), (0, −1), (1, 3), (2, −47), (3, −289), (4, −933)

**8.** (−6, 744), (−4, 154), (−2, 4), (0, −6), (2, 16), (4, 154), (6, 684), (8, 2074), (10, 4984)

**9.** (−2, 968), (−1, 422), (0, 142), (1, 26), (2, −4), (3, −2), (4, 2), (5, 2), (6, 16)

**10.** (1, 0), (2, 6), (3, 2), (4, 6), (5, 12), (6, −10), (7, −114), (8, −378), (9, −904)

4 MTR **11. ERROR ANALYSIS** Describe and correct the error in writing a cubic function whose graph passes through the given points.

(−6, 0), (1, 0), (3, 0), (0, 54)

$$54 = a(0 - 6)(0 + 1)(0 + 3)$$

$$54 = -18a$$

$$-3 = a$$

$$f(x) = -3(x - 6)(x + 1)(x + 3)$$

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4 MTR **12. MAKING AN ARGUMENT** Is it possible to determine the degree of a polynomial function given only the first differences? Explain your reasoning.

7 MTR **13. MODELING REAL LIFE** The table shows the total U.S. biomass energy consumptions $y$ (in trillions of British thermal units, or Btus) $t$ years after 2000. Find a model for the data. Use the model to estimate the total U.S. biomass energy consumption in 2017. *(See Example 3.)*

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 3008 | 2622 | 2701 | 2806 | 3008 |

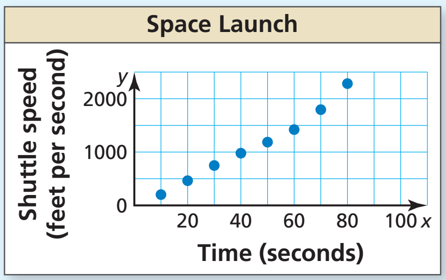
| $t$ | 5 | 10 | 16 | 18 | 19 |
|---|---|---|---|---|---|
| $y$ | 3114 | 4506 | 5015 | 5130 | 4985 |

7 MTR **14. MODELING REAL LIFE** The data in the table show the average speeds $y$ (in miles per hour) of a pontoon boat for several different engine speeds $x$ (in hundreds of revolutions per minute, or RPMs). Find a model for the data. Use the model to estimate the average speed of the pontoon boat when the engine speed is 2800 RPMs.

| $x$ | 10 | 20 | 25 | 30 | 45 | 55 |
|---|---|---|---|---|---|---|
| $y$ | 4.5 | 8.9 | 13.8 | 18.9 | 29.9 | 37.7 |

**15. WRITING** Explain why you cannot always use finite differences to find models for real-life data sets.

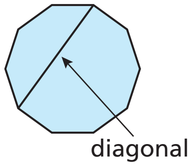
**16. HOW DO YOU SEE IT?**
The graph shows typical speeds $y$ (in feet per second) of a space shuttle $x$ seconds after it is launched.




**a.** Do the data appear to be best represented by a *linear*, *quadratic*, or *cubic* function? Explain.

**b.** Which $n$th differences should be constant for the function in part (a)? Explain.

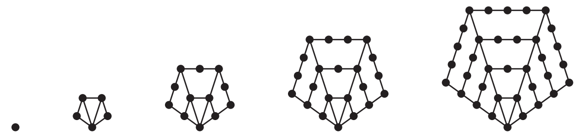
**17. REASONING** The table shows the numbers of diagonals for polygons with $n$ sides. Find a polynomial function that fits the data. Determine the number of diagonals in the decagon shown.




| | | | | | | |
|---|---|---|---|---|---|---|
| **Sides, $n$** | 3 | 4 | 5 | 6 | 7 | 8 |
| **Diagonals, $d$** | 0 | 2 | 5 | 9 | 14 | 20 |

**18. THOUGHT PROVOKING**
Write a polynomial function that has constant fourth differences of $-2$. Justify your answer.

**19. PATTERNS** The figures illustrate the first five pentagonal numbers, where the $n$th pentagonal number is equal to the number of dots in the $n$th figure. Determine the degree of the polynomial function that fits the data. Then find the 10th pentagonal number.



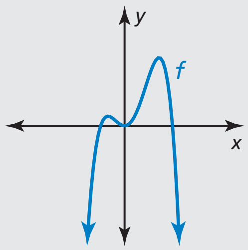
**20. STRUCTURE** Substitute the expressions $z$, $z + 1$, $z + 2, \ldots, z + 5$ for $x$ in the function $f(x) = ax^3 + bx^2 + cx + d$ to generate six equally spaced ordered pairs. Then show that the third differences are constant.

## REVIEW & REFRESH

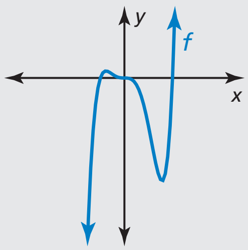
WATCH

**In Exercises 21 and 22, use the graph to describe the degree and leading coefficient of $f$.**

**21.**




**22.**




**23.** Let the graph of $g$ be a translation 2 units left and 5 units down, followed by a reflection in the $x$-axis of the graph of $f(x) = (x + 1)^3 + 3$. Write a rule for $g$.

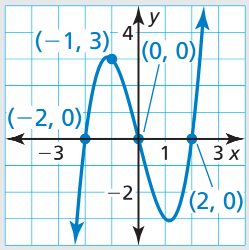
**CHOOSE A METHOD** **In Exercises 24 and 25, solve the system using any method. Explain your choice of method.**

**24.** $y = -2x + 5$
$x^2 + y = 5$

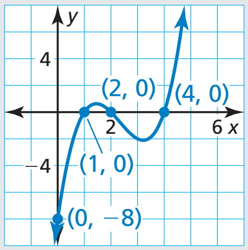
**25.** $x^2 - 3x - y = -3$
$3x^2 - 8x - y = -5$

**In Exercises 26 and 27, write a cubic function whose graph is shown.**

**26.**




**27.**




**28.** Write a quadratic function in standard form whose graph passes through $(-1, 0)$, $(3, -4)$, and $(4, 0)$.

**29.** Find the zeros of $g(x) = 4x^4 - 20x^2 + 16$. Then sketch a graph of $g$.

**30. MODELING REAL LIFE** The table shows the total numbers $y$ of species added to the Endangered Species Act $x$ years since 2010. Use technology to find an equation of the line of best fit. Interpret the slope and $y$-intercept in this situation.

| Years since 2010, $x$ | Species added, $y$ |
|---|---|
| 0 | 54 |
| 1 | 73 |
| 2 | 124 |
| 3 | 213 |
| 4 | 279 |
| 5 | 310 |
| 6 | 384 |
| 7 | 395 |

**In Exercises 31–34, solve the equation.**

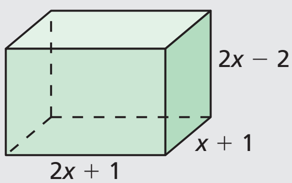
**31.** $x^2 - 6 = 30$

**32.** $5x^2 - 38 = 187$

**33.** $2x^2 + 3x = -3x^2 + 1$

**34.** $4x - 20 = x^2$

**35.** Write an expression for the volume of the rectangular prism as a polynomial in standard form.




**36.** Graph $f(x) = -x^4 + 4x^3 - 8x$. Identify the $x$-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

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