1.2 Transformations of Linear and Absolute Value Functions

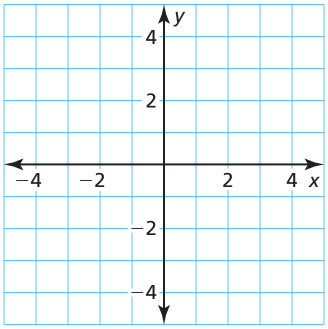
Learning Target: Understand transformations of linear and absolute value functions.

Success Criteria:
- I can write functions that represent transformations of linear and absolute value functions.
- I can identify transformations of linear and absolute value functions.
- I can explain how translations, reflections, stretches, and shrinks affect graphs of functions.

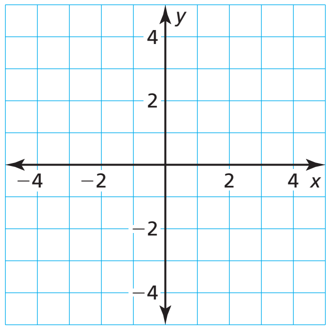
EXPLORE IT! Transforming the Parent Absolute Value Function

Work with a partner. For parts (a)–(d), graph the function for several values of k, h, or a. Then describe how the value of k, h, or a affects the graph.

a. $y = |x| + k$

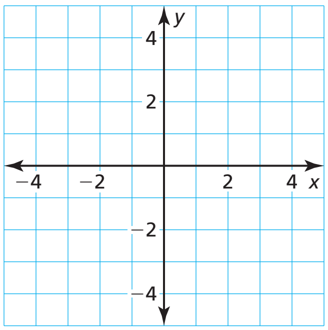


b. $y = |x - h|$

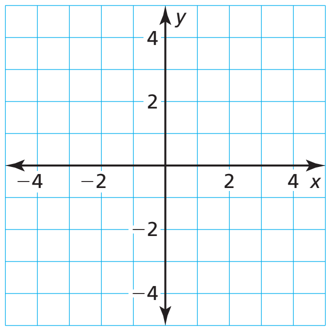


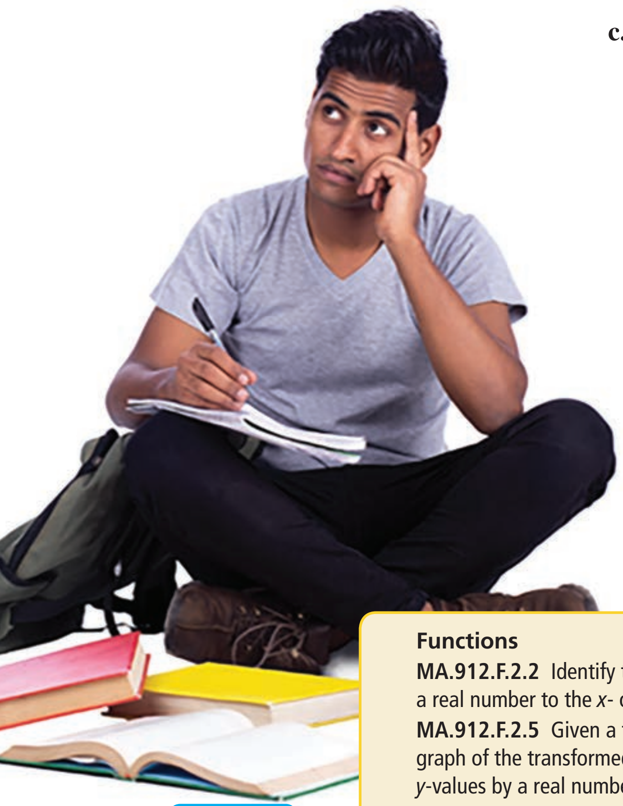
4 MTR CONSTRUCT AN ARGUMENT
In parts (c) and (d), how does a affect the graph when $a = -1$? Explain why this occurs.

c. $y = a \cdot |x|$



d. $y = |a \cdot x|$





e. Let f be the parent absolute value function. How do the graphs compare to the graph of f?

i. $y = f(x) + k$ **ii.** $y = f(x - h)$

iii. $y = a \cdot f(x)$ **iv.** $y = f(a \cdot x)$

Functions
MA.912.F.2.2 Identify the effect on the graph of a given function of two or more transformations defined by adding a real number to the *x*- or *y*-values or multiplying the *x*- or *y*-values by a real number.
MA.912.F.2.5 Given a table, equation or graph that represents a function, create a corresponding table, equation or graph of the transformed function defined by adding a real number to the *x*- or *y*-values or multiplying the *x*- or *y*-values by a real number.
Also MA.912.F.2.3

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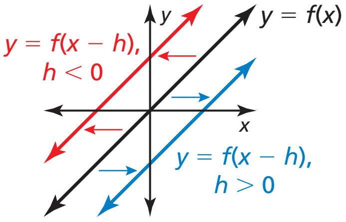
Transformations of Functions

You can use function notation to represent transformations of graphs of functions.

KEY IDEAS

Horizontal Translations

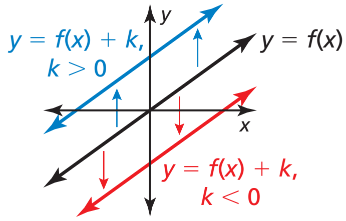
The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$, where $h \neq 0$.



Subtracting h from the *inputs* before evaluating the function shifts the graph left when $h < 0$ and right when $h > 0$.

Vertical Translations

The graph of $y = f(x) + k$ is a vertical translation of the graph of $y = f(x)$, where $k \neq 0$.



Adding k to the *outputs* shifts the graph down when $k < 0$ and up when $k > 0$.

EXAMPLE 1 Writing Translations of Functions

WATCH

Let $f(x) = 2x + 1$.

a. Write a function g whose graph is a translation 3 units down of the graph of f.

b. Write a function h whose graph is a translation 2 units left of the graph of f.

SOLUTION

a. A translation 3 units down is a vertical translation that adds -3 to each output value.

$$\begin{aligned} g(x) &= f(x) + (-3) && \text{Add } -3 \text{ to the output.} \\ &= 2x + 1 + (-3) && \text{Substitute } 2x + 1 \text{ for } f(x). \\ &= 2x - 2 && \text{Simplify.} \end{aligned}$$

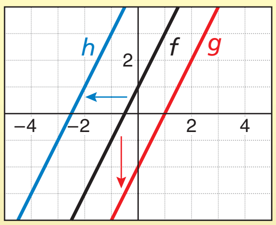
▶ The translated function is $g(x) = 2x - 2$.

b. A translation 2 units left is a horizontal translation that subtracts -2 from each input value.

$$\begin{aligned} h(x) &= f(x - (-2)) && \text{Subtract } -2 \text{ from the input.} \\ &= f(x + 2) && \text{Add the opposite.} \\ &= 2(x + 2) + 1 && \text{Replace } x \text{ with } x + 2 \text{ in } f(x). \\ &= 2x + 5 && \text{Simplify.} \end{aligned}$$

▶ The translated function is $h(x) = 2x + 5$.

Check



SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Write a function g whose graph represents the indicated transformation of the graph of f. Use technology to check your answer.

1. $f(x) = 3x$; translation 5 units up

2. $f(x) = |x| - 3$; translation 4 units right

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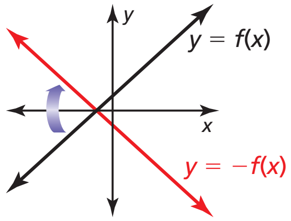
KEY IDEAS

Reflections in the x-Axis

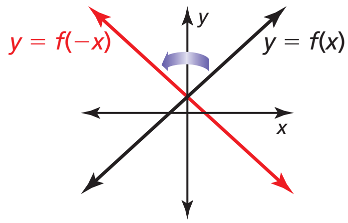
The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.



Multiplying the *outputs* by -1 changes their signs.

Reflections in the y-Axis

The graph of $y = f(-x)$ is a reflection in the y-axis of the graph of $y = f(x)$.



Multiplying the *inputs* by -1 changes their signs.

STUDY TIP

When you reflect a graph in a line, the graphs are symmetric about that line.

EXAMPLE 2 Writing Reflections of Functions

WATCH

Let $f(x) = |x + 3| + 1$.

a. Write a function g whose graph is a reflection in the x-axis of the graph of f.

b. Write a function h whose graph is a reflection in the y-axis of the graph of f.

SOLUTION

a. A reflection in the x-axis changes the sign of each output value.

$g(x) = -f(x)$	Multiply the output by -1.
$= -(\|x + 3\| + 1)$	Substitute $\|x + 3\| + 1$ for $f(x)$.
$= -\|x + 3\| - 1$	Distributive Property

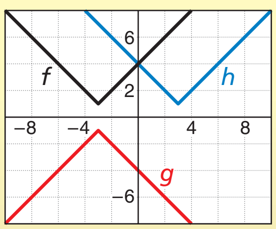
▶ The reflected function is $g(x) = -|x + 3| - 1$.

b. A reflection in the y-axis changes the sign of each input value.

$h(x) = f(-x)$	Multiply the input by -1.
$= \|-x + 3\| + 1$	Replace x with $-x$ in $f(x)$.
$= \|-(x - 3)\| + 1$	Factor out -1.
$= \|-1\| \cdot \|x - 3\| + 1$	Product Property of Absolute Value
$= \|x - 3\| + 1$	Simplify.

▶ The reflected function is $h(x) = |x - 3| + 1$.

Check



SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Write a function g whose graph represents the indicated transformation of the graph of f. Use technology to check your answer.

3. $f(x) = -|x + 2| - 1$; reflection in the x-axis

4. $f(x) = \frac{1}{2}x + 1$; reflection in the y-axis

4 MTR **5. WHICH ONE DOESN'T BELONG?** Let $f(x) = x - 1$ and $g(x) = x + 1$. Which function does *not* belong with the other three? Explain your reasoning.

$h(x) = -f(x)$ $\quad$ $h(x) = f(-x)$ $\quad$ $h(x) = g(-x)$ $\quad$ $h(x) = 1 - x$

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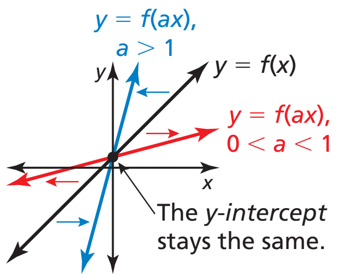
In the previous section, you learned that vertical stretches and shrinks transform graphs. You can also use *horizontal* stretches and shrinks to transform graphs.

KEY IDEAS

Horizontal Stretches and Shrinks

The graph of $y = f(ax)$ is a horizontal stretch or shrink by a factor of $\frac{1}{a}$ of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.

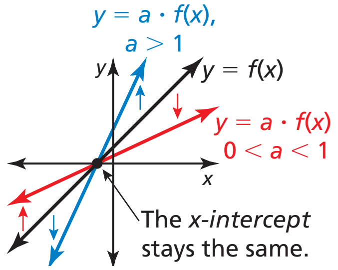
Multiplying the *inputs* by a before evaluating the function stretches the graph horizontally (away from the y-axis) when $0 < a < 1$, and shrinks the graph horizontally (toward the y-axis) when $a > 1$.



Vertical Stretches and Shrinks

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.

Multiplying the *outputs* by a stretches the graph vertically (away from the x-axis) when $a > 1$, and shrinks the graph vertically (toward the x-axis) when $0 < a < 1$.



STUDY TIP

The graphs of $y = f(-ax)$ and $y = -a \cdot f(x)$ represent a stretch or shrink *and* a reflection in the x- or y-axis of the graph of $y = f(x)$.

EXAMPLE 3 Writing Stretches and Shrinks of Functions

WATCH

Let $f(x) = |x - 3| - 5$. Write (a) a function g whose graph is a horizontal shrink of the graph of f by a factor of $\frac{1}{3}$, and (b) a function h whose graph is a vertical stretch of the graph of f by a factor of 2.

SOLUTION

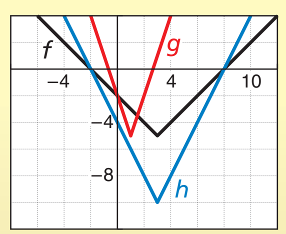
a. A horizontal shrink by a factor of $\frac{1}{3}$ multiplies each input value by 3.

$g(x) = f(3x)$ — Multiply the input by 3.

$= |3x - 3| - 5$ — Replace x with $3x$ in $f(x)$.

▶ The transformed function is $g(x) = |3x - 3| - 5$.

Check



b. A vertical stretch by a factor of 2 multiplies each output value by 2.

$h(x) = 2 \cdot f(x)$ — Multiply the output by 2.

$= 2 \cdot (|x - 3| - 5)$ — Substitute $|x - 3| - 5$ for $f(x)$.

$= 2|x - 3| - 10$ — Distributive Property

▶ The transformed function is $h(x) = 2|x - 3| - 10$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Write a function g whose graph represents the indicated transformation of the graph of f. Use technology to check your answer.

6. $f(x) = 4x + 2$; horizontal stretch by a factor of 2

7. $f(x) = |x| - 3$; vertical shrink by a factor of $\frac{1}{3}$

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EXAMPLE 4 Describing a Transformation

x	-3	-2	-1	0
$f(x)$	10	6	2	6
$g(x)$	5	3	1	3

The table represents two absolute value functions f and g. Describe the transformation from the graph of f to the graph of g.

SOLUTION

To determine the transformation, compare values of $f(x)$ and $g(x)$.

x	-3	-2	-1	0
$f(x)$	10	6	2	6
	$\times \frac{1}{2}$	$\times \frac{1}{2}$	$\times \frac{1}{2}$	$\times \frac{1}{2}$
$g(x)$	5	3	1	3

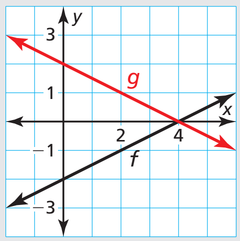
For each input, $g(x)$ is $\frac{1}{2}$ times $f(x)$. So, $g(x) = \frac{1}{2}f(x)$.

▶ The graph of g is a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

8. Use the graph to describe the transformation from the graph of f to the graph of g.



Combinations of Transformations

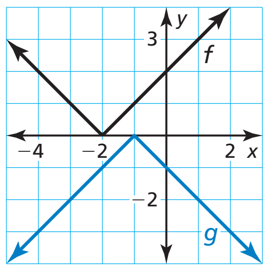
EXAMPLE 5 Describing a Combination of Transformations

Describe the transformation of $f(x) = |x + 2|$ represented by $g(x) = -f(x - 1)$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = a \bullet f(x - h)$, where $a = -1$ and $h = 1$.

▶ So, the graph of g is a reflection in the x-axis and a horizontal translation 1 unit right of the graph of f.



5 MTR USE STRUCTURE
Does the order of the transformations matter in this situation? Justify your answer.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

9. Describe the transformation of $f(x) = x - 4$ represented by $g(x) = 2f(x) + 5$. Then graph each function.

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You can write a function that represents a series of transformations on the graph of another function by applying the transformations one at a time in the stated order.

EXAMPLE 6 Writing a Combination of Transformations

Let the graph of g be a vertical shrink by a factor of 0.25 followed by a translation 3 units up of the graph of $f(x) = x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the vertical shrink of f.

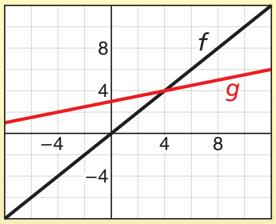
$$h(x) = 0.25 \cdot f(x) \qquad \text{Multiply the output by 0.25.}$$
$$= 0.25x \qquad \text{Substitute } x \text{ for } f(x).$$

Step 2 Then write a function g that represents the translation of h.

$$g(x) = h(x) + 3 \qquad \text{Add 3 to the output.}$$
$$= 0.25x + 3 \qquad \text{Substitute } 0.25x \text{ for } h(x).$$

▶ The transformed function is $g(x) = 0.25x + 3$.

Check



7 MTR EXAMPLE 7 Modeling Real Life

You design a computer game. Your revenue (in dollars) for x downloads is given by $f(x) = 2x$, and your profit is \$50 less than 90% of the revenue. What is your profit for 100 downloads?

SOLUTION

1. **Understand the Problem** You are given a function that represents your revenue and a verbal statement that represents your profit. You are asked to find your profit for 100 downloads.
2. **Make a Plan** Write a function p that represents your profit. Then use this function to find the profit for 100 downloads.
3. **Solve and Check** profit = 90% • revenue − 50

$$p(x) = 0.9 \cdot f(x) - 50$$

Vertical shrink by a factor of 0.9

Translation 50 units down

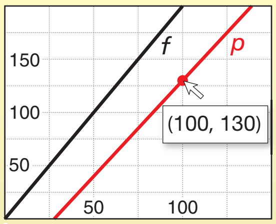
$$= 0.9 \cdot 2x - 50 \qquad \text{Substitute } 2x \text{ for } f(x).$$
$$= 1.8x - 50 \qquad \text{Simplify.}$$

To find the profit for 100 downloads, evaluate p when $x = 100$.

$$p(100) = 1.8(100) - 50 = 130$$

▶ Your profit is \$130 for 100 downloads.

Look Back The vertical shrink decreases the slope, and the translation shifts the graph 50 units down. So, the graph of p is below and not as steep as the graph of f.



SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

10. Let the graph of g be a translation 6 units down followed by a reflection in the x-axis of the graph of $f(x) = |x|$. Write a rule for g. Use technology to check your answer.

11. **WHAT IF?** In Example 7, your revenue function is $f(x) = 3x$. How does this affect your profit for 100 downloads?

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1.2 Practice WITH CalcChat® AND CalcView®

In Exercises 1–6, write a function g whose graph represents the indicated transformation of the graph of f. Use technology to check your answer. *(See Example 1.)*

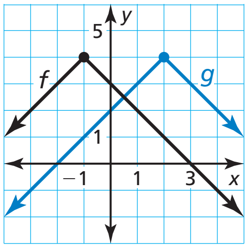
1. $f(x) = x - 5$; translation 4 units left

2. $f(x) = x + 2$; translation 2 units right

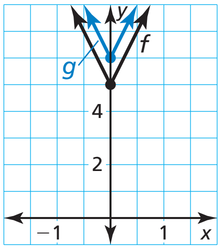
3. $f(x) = |4x + 3| + 2$; translation 2 units down

4. $f(x) = 2|x| - 9$; translation 6 units up

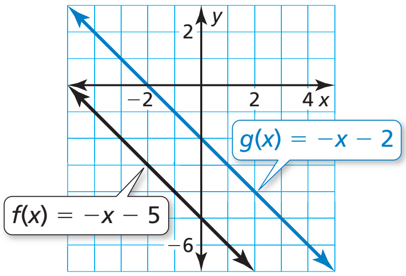
5. $f(x) = 4 - |x + 1|$



6. $f(x) = |4x| + 5$



7. WRITING Describe the translation from the graph of f to the graph of g in two different ways.



8. PROBLEM SOLVING You start a photography business. The function $f(x) = 4000x$ represents your expected total net income (in dollars) after x weeks. Before you start, you incur an expense of \$12,000. What transformation of f is necessary to model this situation? How many weeks will it take to pay off the extra expense?

In Exercises 9–14, write a function g whose graph represents the indicated transformation of the graph of f. Use technology to check your answer. *(See Example 2.)*

9. $f(x) = -5x + 2$; reflection in the x-axis

10. $f(x) = \frac{1}{2}x - 3$; reflection in the x-axis

11. $f(x) = |6x| - 2$; reflection in the y-axis

12. $f(x) = |2x - 1| + 3$; reflection in the y-axis

13. $f(x) = -3 + |x - 11|$; reflection in the y-axis

14. $f(x) = -x + 1$; reflection in the y-axis

In Exercises 15–22, write a function g whose graph represents the indicated transformation of the graph of f. Use technology to check your answer. *(See Example 3.)*

15. $f(x) = x + 2$; vertical stretch by a factor of 5

16. $f(x) = 2x + 6$; vertical shrink by a factor of $\frac{1}{2}$

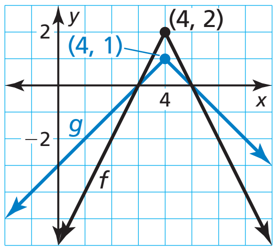
17. $f(x) = |2x| + 4$; horizontal shrink by a factor of $\frac{1}{2}$

18. $f(x) = |x + 3|$; horizontal stretch by a factor of 4

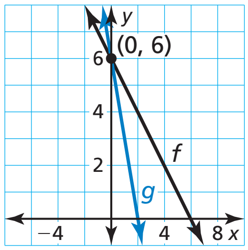
19. $f(x) = x - 3$; horizontal stretch by a factor of 2

20. $f(x) = |x + 1| - 1$; vertical stretch by a factor of 3

21. $f(x) = -2|x - 4| + 2$



22. $f(x) = 6 - x$



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In Exercises 23–26, create a table that represents the transformation of f given by g.

23. $g(x) = f(x - 4)$

x	-1	0	1	2
$f(x)$	-4	-3	-2	-1

24. $g(x) = f(x) + 5$

x	-7	-5	-3	-1
$f(x)$	3	1	1	3

25. $g(x) = 3f(x)$

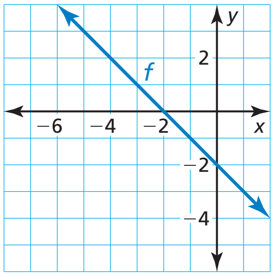
x	-3	-1	1	3
$f(x)$	-5	-1	3	7

26. $g(x) = -f(x)$

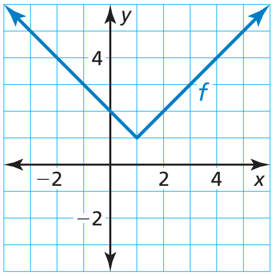
x	-2	0	2	4
$f(x)$	10	2	6	14

In Exercises 27–30, graph g.

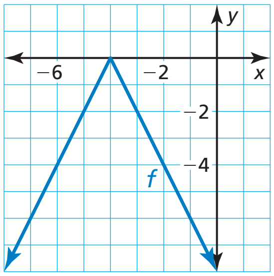
27. $g(x) = f(x + 1)$



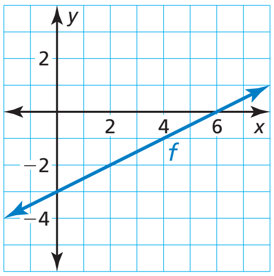
28. $g(x) = f(x) - 6$



29. $g(x) = 0.5f(x)$



30. $g(x) = f(4x)$



In Exercises 31–38, describe the transformations from the graph of f to the graph of g. *(See Example 4.)*

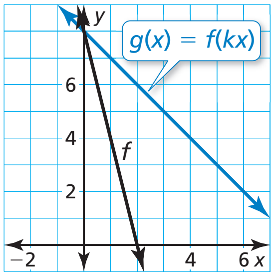
31.

x	4	5	6	7
$f(x)$	1	0	1	2
$g(x) = f(x) - k$	-3	-4	-3	-2

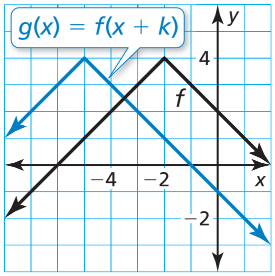
32.

x	-2	-1	0	1
$f(x)$	1	4	7	10
$g(x) = -f(x)$	-1	-4	-7	-10

33.



34.



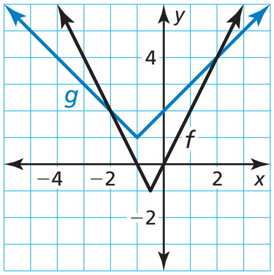
35.

x	-4	-2	0	2
$f(x)$	4	5	6	7
$g(x)$	8	10	12	14

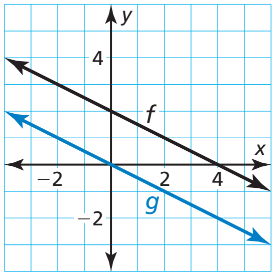
36.

x	3	4	5	6
$f(x)$	0	-1	0	1
$g(x)$	2	1	2	3

37.



38.



In Exercises 39–42, describe the transformation of f represented by g. Then graph each function. *(See Example 5.)*

39. $f(x) = |x - 9|$; $g(x) = f(x + 5) - 1$

40. $f(x) = \frac{1}{2}x - 4$; $g(x) = f(x - 2) + 7$

41. $f(x) = 3x + 11$; $g(x) = -f\left(\frac{1}{3}x\right)$

42. $f(x) = |x + 2| - 5$; $g(x) = 3f(x) - 4$

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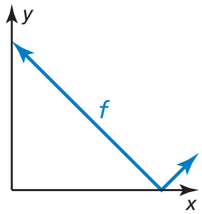
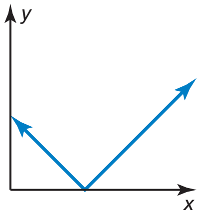
5 MTR **STRUCTURE** **In Exercises 43–46, match the graph of the transformation of f with the correct equation shown. Explain your reasoning.**



43.



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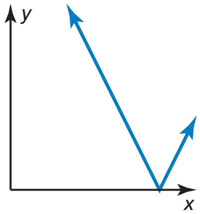

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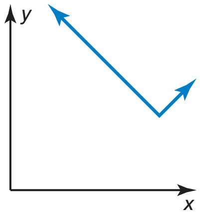
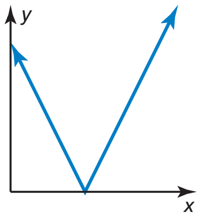


46.



A. $y = 2f(x)$ **C.** $y = f(x + 2)$

B. $y = f(2x)$ **D.** $y = f(x) + 2$

In Exercises 47–52, write a function g whose graph represents the indicated transformations of the graph of f. *(See Example 6.)*

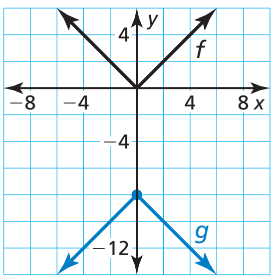
47. $f(x) = x$; vertical stretch by a factor of 2 followed by a translation 1 unit up

48. $f(x) = x$; translation 3 units down followed by a vertical shrink by a factor of $\frac{1}{3}$

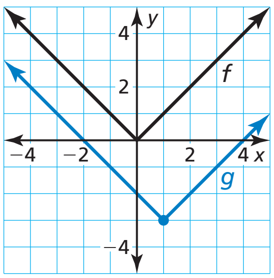
▶ **49.** $f(x) = |x|$; translation 2 units right followed by a horizontal stretch by a factor of 2

50. $f(x) = |x|$; reflection in the y-axis followed by a translation 3 units right

51. $f(x) = |x|$



52. $f(x) = |x|$



4 MTR **ERROR ANALYSIS** **In Exercises 53 and 54, identify and correct the error in writing the function g whose graph represents the indicated transformations of the graph of f.**

53.

> ✗ $f(x) = |x|$; translation 3 units right followed by a translation 2 units up
>
> $g(x) = |x + 3| + 2$

54.

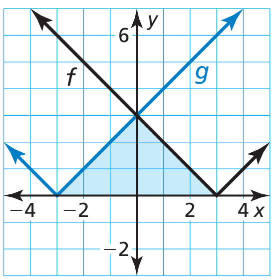
> ✗ $f(x) = x$; translation 6 units down followed by a vertical stretch by a factor of 5
>
> $g(x) = 5x - 6$

7 MTR **55.** **MODELING REAL LIFE** The cost (in dollars) of a car ride from a ride-sharing company during regular hours is modeled by $f(x) = 2.30x$, where x is the number of miles driven. The cost of a ride during high-demand hours, including a tip, is \$5 more than 120% the cost during regular hours. What is the cost of a 6-mile ride during high-demand hours? *(See Example 7.)*

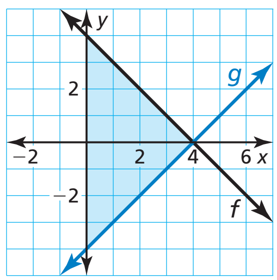
7 MTR **56.** **MODELING REAL LIFE** Recently, bookstore sales have been declining. The sales (in billions of dollars) can be modeled by the function $f(t) = -\frac{1}{4}t + 11.3$, where t is the number of years since 2016. Transform the graph of f to model sales that decrease at twice this rate. Explain how this affects bookstore sales in 2024.

5 MTR **CONNECTING CONCEPTS** **In Exercises 57 and 58, describe the transformation of the graph of f to the graph of g. Then find the area of the shaded triangle.**

57. $f(x) = |x - 3|$



58. $f(x) = -x + 4$



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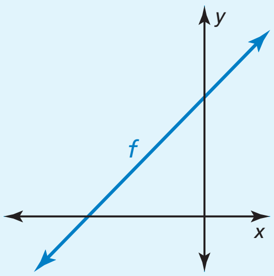
59. REASONING Give an example of a function and a series of transformations where the order of the transformations does not matter. Then give an example where the order does matter. Explain.

60. HOW DO YOU SEE IT? Consider the graph of $f(x) = mx + b$. Describe the effect each transformation has on the slope of the line and the intercepts of the graph.



a. Reflect the graph of f in the y-axis.

b. Shrink the graph of f vertically by a factor of $\frac{1}{3}$.

c. Stretch the graph of f horizontally by a factor of 2.

61. REASONING Complete the function $g(x) = \square\,|x - \square| + \square$ so that g is a reflection in the x-axis followed by a translation one unit left and one unit up of the graph of $f(x) = 2|x - 2| + 1$. Explain your reasoning.

62. THOUGHT PROVOKING

Let $f(x) = a|x - h| + k$ and $g(x) = -|x - j| - \frac{k}{a}$, where a, h, j, and k are positive integers. Describe the transformations of the graph of f to the graph of g in terms of a, h, j, and k.

63. DIG DEEPER The functions $f(x) = mx + b$ and $g(x) = mx + c$ represent two parallel lines. Write an expression for the horizontal translation of the graph of f to the graph of g.

REVIEW & REFRESH

WATCH

In Exercises 64 and 65, evaluate the function for the given value of x.

64. $f(x) = x + 4$; $x = 3$

65. $f(x) = -2x - 2$; $x = -1$

In Exercises 66 and 67, make a scatter plot of the data. Then describe the relationship between the data.

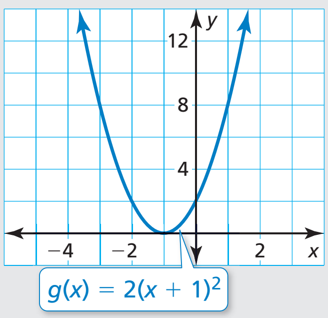
66.

x	8	10	11	12	15
$f(x)$	4	9	10	12	12

67.

x	2	5	6	10	13
$f(x)$	22	13	15	12	6

68. Identify the function family to which g belongs. Compare the graph of the function to the graph of its parent function.



In Exercises 69–72, solve the system using any method. Explain your choice of method.

69. $3x - 2y = -15$
$4x + 2y = 8$

70. $y = \frac{2}{3}x - 4$
$y = \frac{4}{3}x + 2$

71. $x = -4y + 7$
$-2y + 3x = 9$

72. $2.5x - 2.5y = 10$
$-5x + 5y = -15$

7 MTR **73. MODELING REAL LIFE** The function $f(x) = -1.5x + 50$ represents the amount (in pounds) of dog food in a bag after x days.

a. Graph the function and find its domain and range.

b. Interpret the slope and the intercepts of the graph.

In Exercises 74–77, graph the function. Identify the domain and range.

74. $f(x) = \frac{3}{2}x^2$

75. $g(x) = -x^2 + 5$

76. $p(x) = 3(x - 1)^2$

77. $q(x) = -\frac{1}{2}(x + 4)^2 - 6$

In Exercises 78 and 79, write a function g whose graph represents the indicated transformations of the graph of f.

78. $f(x) = x$; translation 2 units down and a horizontal shrink by a factor of $\frac{2}{3}$

79. $f(x) = |x|$; reflection in the x-axis and a vertical stretch by a factor of 4

GO DIGITAL