9.1 Properties of Radicals

Learning Target: Use properties of radicals to write equivalent expressions.

Success Criteria:
- I can use properties of square roots to write equivalent expressions.
- I can use properties of cube roots to write equivalent expressions.
- I can rationalize the denominator of a fraction.
- I can perform operations with radicals.

EXPLORE IT! Performing Operations with Square Roots

Work with a partner.

a. Determine which of the following are general rules by choosing various values for a and b. Use what you know about radicals and rational exponents to justify your answer.

i. $\sqrt{a} + \sqrt{b} \stackrel{?}{=} \sqrt{a + b}$

ii. $\sqrt{a} \cdot \sqrt{b} \stackrel{?}{=} \sqrt{a \cdot b}$

iii. $\sqrt{a} - \sqrt{b} \stackrel{?}{=} \sqrt{a - b}$

iv. $\dfrac{\sqrt{a}}{\sqrt{b}} \stackrel{?}{=} \sqrt{\dfrac{a}{b}}$

1 MTR ANALYZE A PROBLEM
What types of values of *a* and *b* make the expressions easier to evaluate?

b. Explain in your own words how to add, subtract, multiply, and divide square roots.

c. Consider the list of rational and irrational numbers shown. Add two more rational numbers and two more irrational numbers to the list.

$2 \qquad \dfrac{1}{4} \qquad 0 \qquad \sqrt{3} \qquad -\sqrt{3} \qquad \dfrac{1}{\sqrt{3}} \qquad \pi$

Experiment with sums and products of two numbers from the list to determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

i. The sum of two rational numbers is rational.

ii. The sum of a rational number and an irrational number is irrational.

iii. The sum of two irrational numbers is irrational.

iv. The product of two rational numbers is rational.

v. The product of a nonzero rational number and an irrational number is irrational.

vi. The product of two irrational numbers is irrational.

Number Sense and Operations
MA.912.NSO.1.4 Apply previous understanding of operations with rational numbers to add, subtract, multiply and divide numerical radicals.

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Vocabulary

radical expression, *p. 470*
simplest form of a radical, *p. 470*
rationalizing the denominator, *p. 471*
like radicals, *p. 472*

Using Properties of Radicals

A **radical expression** is an expression that contains a radical. An expression involving a radical with index n is in **simplest form** when these three conditions are met:

- No radicands have perfect nth powers as factors other than 1.
- No radicands contain fractions.
- No radicals appear in the denominator of a fraction.

You can use the properties below to simplify radical expressions involving square roots.

KEY IDEAS

Product Property of Square Roots

Words The square root of a product equals the product of the square roots of the factors.

Numbers $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

Algebra $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a \geq 0$ and $b \geq 0$

Quotient Property of Square Roots

Words The square root of a quotient equals the quotient of the square roots of the numerator and denominator.

Numbers $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$

Algebra $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, where $a \geq 0$ and $b > 0$

There can be more than one way to factor a radicand. An efficient method is to find the greatest perfect square factor.

EXAMPLE 1 Using Properties of Square Roots

a. $\sqrt{108} = \sqrt{36 \cdot 3}$ Factor using the greatest perfect square factor.

$= \sqrt{36} \cdot \sqrt{3}$ Product Property of Square Roots

$= 6\sqrt{3}$ Simplify.

b. $\sqrt{\frac{15}{64}} = \frac{\sqrt{15}}{\sqrt{64}}$ Quotient Property of Square Roots

$= \frac{\sqrt{15}}{8}$ Simplify.

c. $\sqrt{\frac{198}{2}} = \sqrt{99}$ Divide.

$= \sqrt{9 \cdot 11}$ Factor using the greatest perfect square factor.

$= \sqrt{9} \cdot \sqrt{11}$ Product Property of Square Roots

$= 3\sqrt{11}$ Simplify.

You can extend the Product and Quotient Properties of Square Roots to other radicals, such as cube roots. When using these *properties of cube roots*, the radicands can contain negative numbers.

EXAMPLE 2 **Using Properties of Cube Roots** WATCH

STUDY TIP
To write a cube root in simplest form, find factors of the radicand that are perfect cubes.

a. $\sqrt[3]{-128} = \sqrt[3]{-64 \cdot 2}$ Factor using the greatest perfect cube factor.

$= \sqrt[3]{-64} \cdot \sqrt[3]{2}$ Product Property of Cube Roots

$= -4\sqrt[3]{2}$ Simplify.

b. $\sqrt[3]{\dfrac{9}{216}} = \dfrac{\sqrt[3]{9}}{\sqrt[3]{216}}$ Quotient Property of Cube Roots

$= \dfrac{\sqrt[3]{9}}{6}$ Simplify.

SELF-ASSESSMENT 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Simplify the expression.

1. $\sqrt{24}$ **2.** $-\sqrt{\dfrac{17}{100}}$ **3.** $\sqrt[3]{54}$ **4.** $\sqrt[3]{\dfrac{27}{8}}$

Rationalizing the Denominator

When a radical is in the denominator of a fraction, you can multiply the fraction by an appropriate form of 1 to eliminate the radical from the denominator. This process is called **rationalizing the denominator**.

EXAMPLE 3 **Rationalizing the Denominator** WATCH

STUDY TIP
Rationalizing the denominator works because you multiply the numerator and denominator by the same nonzero number *a*, which is the same as multiplying by $\frac{a}{a}$, or 1.

a. $\dfrac{5}{\sqrt{6}} = \dfrac{5}{\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}}$ Multiply by $\dfrac{\sqrt{6}}{\sqrt{6}}$.

$= \dfrac{5\sqrt{6}}{6}$ Simplify.

b. $\dfrac{2}{\sqrt[3]{9}} = \dfrac{2}{\sqrt[3]{9}} \cdot \dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$ Multiply by $\dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$.

$= \dfrac{2\sqrt[3]{3}}{\sqrt[3]{27}}$ Product Property of Cube Roots

$= \dfrac{2\sqrt[3]{3}}{3}$ Simplify.

SELF-ASSESSMENT 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Simplify the expression.

5. $\dfrac{1}{\sqrt{5}}$ **6.** $\dfrac{2}{\sqrt[3]{25}}$ **7.** $\dfrac{\sqrt{10}}{\sqrt{3}}$ **8.** $\dfrac{5}{\sqrt[3]{32}}$

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Performing Operations with Radicals

Radicals with the same index and radicand are called **like radicals**. You can add and subtract like radicals the same way you combine like terms by using the Distributive Property.

EXAMPLE 4 Adding and Subtracting Radicals

WATCH

a. $5\sqrt{7} - 8\sqrt{7} = (5 - 8)\sqrt{7}$ — Distributive Property

$= -3\sqrt{7}$ — Subtract.

b. $10\sqrt{5} + \sqrt{20} = 10\sqrt{5} + 2\sqrt{5}$ — Simplify.

$= (10 + 2)\sqrt{5}$ — Distributive Property

$= 12\sqrt{5}$ — Add.

c. $\sqrt[3]{24} + \sqrt[3]{192} = 2\sqrt[3]{3} + 4\sqrt[3]{3}$ — Simplify.

$= (2 + 4)\sqrt[3]{3}$ — Distributive Property

$= 6\sqrt[3]{3}$ — Add.

STUDY TIP

Do not assume that radicals with different radicands cannot be added or subtracted. Always check to see whether you can simplify the radicals. In some cases, the radicals can be written as like radicals.

EXAMPLE 5 Multiplying Radicals

WATCH

a. $\sqrt{6} \cdot \sqrt{21} = \sqrt{6 \cdot 21}$ — Product Property of Square Roots

$= \sqrt{126}$ — Multiply.

$= 3\sqrt{14}$ — Simplify.

b. $\sqrt[3]{9} \cdot \sqrt[3]{5} = \sqrt[3]{9 \cdot 5}$ — Product Property of Cube Roots

$= \sqrt[3]{45}$ — Multiply.

c. $\sqrt[3]{4} \cdot \left(-2\sqrt[3]{3}\right) = -2\left(\sqrt[3]{4} \cdot \sqrt[3]{3}\right)$ — Commutative Property of Multiplication

$= -2\sqrt[3]{4 \cdot 3}$ — Product Property of Cube Roots

$= -2\sqrt[3]{12}$ — Multiply.

1 MTR HELP A CLASSMATE

In part (a), help a classmate use properties of radicals to verify that $\sqrt{126} = 3\sqrt{14}$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Simplify the expression.

9. $3\sqrt{2} + 10\sqrt{2}$

10. $4\sqrt{7} - 6\sqrt{63}$

11. $\sqrt{11} \cdot \sqrt{2}$

12. $-\sqrt[3]{10} \cdot \left(-\sqrt[3]{4}\right)$

4 MTR **13. WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$-\frac{1}{3}\sqrt{6}$ $\quad 6\sqrt{3}$ $\quad \frac{1}{6}\sqrt{3}$ $\quad -3\sqrt{3}$

5 MTR **14. MAKE A PLAN** The product below is equal to 12. Make a plan for how to find the missing value. Then find the value and justify your answer.

$$2\sqrt[3]{-24} \cdot \sqrt[3]{\square}$$

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EXAMPLE 6 Dividing Radicals

a. $\sqrt{10} \div \sqrt{2} = \sqrt{10 \div 2}$ Quotient Property of Square Roots

$= \sqrt{5}$ Divide.

b. $\sqrt[3]{-48} \div \sqrt[3]{6} = \sqrt[3]{-48 \div 6}$ Quotient Property of Cube Roots

$= \sqrt[3]{-8}$ Divide.

$= -2$ Simplify.

c. $\sqrt{160} \div \sqrt{5} = \sqrt{160 \div 5}$ Quotient Property of Square Roots

$= \sqrt{32}$ Divide.

$= 4\sqrt{2}$ Simplify.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Simplify the expression.

15. $\sqrt{26} \div \sqrt{13}$ **16.** $\sqrt[3]{16} \div \sqrt[3]{4}$ **17.** $\sqrt{252} \div \left(-\sqrt{7}\right)$ **18.** $\sqrt[3]{-24} \div 2\sqrt[3]{54}$

Solving Real-Life Problems

7 MTR

EXAMPLE 7 Modeling Real Life

The distance d (in miles) that you can see to the horizon with your eye level h feet above the water is given by $d = \dfrac{\sqrt{3h}}{\sqrt{2}}$. Estimate how far you can see when your eye level is 5 feet above the water.

SOLUTION

Use the equation to find the value of d when $h = 5$.

$d = \dfrac{\sqrt{3(5)}}{\sqrt{2}}$ Substitute 5 for h.

$= \dfrac{\sqrt{15}}{\sqrt{2}}$ Multiply.

$= \dfrac{\sqrt{15}}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$ Multiply by $\dfrac{\sqrt{2}}{\sqrt{2}}$.

$= \dfrac{\sqrt{30}}{2}$ Simplify.

▶ You can see $\dfrac{\sqrt{30}}{2} \approx 2.7$ miles.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

19. WHAT IF? In Example 7, how far can you see when your eye level is 35 feet above the water?

20. The velocity v (in feet per second) of a dropped object is given by $v = \sqrt{64d}$, where d is the distance the object falls before hitting the ground. Panorama Tower in Miami is 868 feet tall. What is the maximum velocity of an object dropped from the top of Panorama Tower?

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9.1 Practice WITH CalcChat® AND CalcView®

In Exercises 1–6, determine whether the expression is in simplest form. If it is not, explain why not.

1. $\sqrt{19}$ **2.** $\sqrt{\frac{1}{7}}$

3. $\sqrt{34}$ **4.** $\frac{5}{\sqrt{2}}$

5. $\frac{3\sqrt{10}}{4}$ **6.** $\frac{5\sqrt{24}}{12}$

In Exercises 7–14, simplify the expression.
(See Example 1.)

7. $\sqrt{20}$ **8.** $\sqrt{32}$

9. $\sqrt{128}$ **10.** $-\sqrt{72}$

11. $\sqrt{\frac{4}{49}}$ **12.** $-\sqrt{\frac{7}{81}}$

13. $-\sqrt{\frac{23}{64}}$ **14.** $\sqrt{\frac{65}{121}}$

In Exercises 15–22, simplify the expression.
(See Example 2.)

15. $\sqrt[3]{16}$ **16.** $\sqrt[3]{-108}$

17. $\sqrt[3]{-64}$ **18.** $-\sqrt[3]{343}$

19. $\sqrt[3]{\frac{6}{-125}}$ **20.** $\sqrt[3]{\frac{8}{27}}$

21. $-\sqrt[3]{\frac{81}{1000}}$ **22.** $\sqrt[3]{\frac{21}{-64}}$

4 MTR **23. ERROR ANALYSIS** Describe and correct the error in writing $\sqrt{72}$ in simplest form.

$$\begin{aligned}\sqrt{72} &= \sqrt{4 \cdot 18}\\ &= \sqrt{4} \cdot \sqrt{18}\\ &= 2\sqrt{18}\end{aligned}$$

4 MTR **24. ERROR ANALYSIS** Describe and correct the error in writing $\sqrt[3]{\frac{128}{125}}$ in simplest form.

$$\begin{aligned}\sqrt[3]{\frac{128}{125}} &= \frac{\sqrt[3]{128}}{125}\\ &= \frac{\sqrt[3]{64 \cdot 2}}{125}\\ &= \frac{\sqrt[3]{64} \cdot \sqrt[3]{2}}{125}\\ &= \frac{4\sqrt[3]{2}}{125}\end{aligned}$$

In Exercises 25–28, write a form of 1 that you can use to rationalize the denominator of the expression.

25. $\frac{4}{\sqrt{6}}$ **26.** $\frac{10}{\sqrt{3}}$

27. $\frac{8}{\sqrt[3]{4}}$ **28.** $\frac{\sqrt[3]{7}}{\sqrt[3]{5}}$

In Exercises 29–34, simplify the expression.
(See Example 3.)

29. $\frac{2}{\sqrt{2}}$ **30.** $\frac{4}{\sqrt{3}}$

31. $\frac{\sqrt{5}}{\sqrt{48}}$ **32.** $\sqrt{\frac{4}{52}}$

33. $\frac{4}{\sqrt[3]{25}}$ **34.** $\frac{\sqrt[3]{2}}{\sqrt[3]{49}}$

In Exercises 35–42, simplify the expression.
(See Example 4.)

35. $2\sqrt{2} + 6\sqrt{2}$ **36.** $5\sqrt{13} - 8\sqrt{117}$

37. $2\sqrt{6} - 5\sqrt{54}$ **38.** $9\sqrt{32} + \sqrt{2}$

39. $\sqrt[3]{-81} + 4\sqrt[3]{3}$ **40.** $6\sqrt[3]{128} - 2\sqrt[3]{2}$

41. $2\sqrt[3]{48} - \sqrt[3]{162}$ **42.** $\sqrt[3]{192} + 5\sqrt[3]{64}$

In Exercises 43–50, simplify the expression.
(See Examples 5 and 6.)

43. $\sqrt{2} \cdot \sqrt{45}$ **44.** $4\sqrt{3} \cdot \sqrt{72}$

45. $7\sqrt[3]{5} \cdot \left(-2\sqrt[3]{15}\right)$ **46.** $\left(-4\sqrt[3]{29}\right) \cdot \left(-3\sqrt[3]{38}\right)$

47. $\sqrt{75} \div \sqrt{3}$ **48.** $3\sqrt{24} \div \sqrt{8}$

49. $12\sqrt[3]{750} \div 4\sqrt[3]{6}$ **50.** $5\sqrt[3]{108} \div \sqrt[3]{-50}$

7 MTR **51. MODELING REAL LIFE** The time t (in seconds) it takes a dropped object to fall h feet is given by $t = \sqrt{\frac{h}{16}}$.

a. Estimate how long it takes an earring to hit the ground when it falls from the roof of the building.

b. Estimate how much sooner the earring hits the ground when it is dropped from two stories (22 feet) below the roof.

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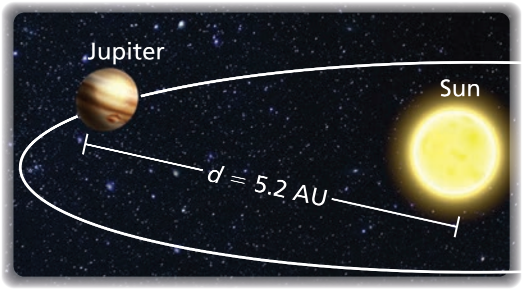
52. MODELING REAL LIFE The orbital period of a planet is the time it takes the planet to travel around the Sun. You can find the orbital period P (in Earth years) using the formula $P = \sqrt{d^3}$, where d is the average distance (in astronomical units, abbreviated AU) of the planet from the Sun. Estimate Jupiter's orbital period.



53. MODELING REAL LIFE The electric current I (in amperes) an appliance uses is given by $I = \sqrt{\frac{P}{R}}$, where P is the power (in watts) and R is the resistance (in ohms). Estimate the current an appliance uses when the power is 147 watts and the resistance is 5 ohms. *(See Example 7.)*

54. PROBLEM SOLVING The radius r of a cylinder is given by

$$r = \sqrt{\frac{V}{\pi h}}$$

where V is the volume and h is the height of the cylinder.

a. Estimate the radius of a cylinder with a volume of 44π cubic centimeters and a height of 7 centimeters.

b. The radius of Cylinder A is 30 inches. Cylinder B has the same volume as Cylinder A but is 9 times taller. What is the radius of Cylinder B?

In Exercises 55–58, evaluate the function for the given value of x. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

55. $h(x) = \sqrt{5x};\ x = 10$

56. $g(x) = \sqrt{3x};\ x = 60$

57. $p(x) = \sqrt[3]{6x};\ x = 4$

58. $k(x) = \sqrt[3]{9x};\ x = -54$

In Exercises 59–62, evaluate the expression when $a = -2$, $b = 8$, and $c = \frac{1}{2}$. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

59. $\sqrt{a^2 + bc}$

60. $-\sqrt{4c - 6ab}$

61. $-\sqrt{2a^2 + b^2}$

62. $\sqrt{b^2 - 4ac}$

63. REASONING Let m be a positive integer. For what values of m will the simplified form of the expression $\sqrt{2^m}$ contain a radical? For what values will it *not* contain a radical? Explain.

64. B.E.S.T. TEST PREP Which expression is equivalent to $2\sqrt[3]{192}$?

Ⓐ $4\sqrt[3]{3}$

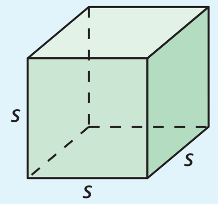
Ⓑ $6\sqrt[3]{3}$

Ⓒ $8\sqrt[3]{3}$

Ⓓ $16\sqrt[3]{3}$

65. REASONING Let a and b be positive numbers. Explain why $\sqrt{ab}$ lies between a and b on a number line. (*Hint:* Let $a < b$ and multiply each side of $a < b$ by a. Then let $a < b$ and multiply each side by b.)

66. HOW DO YOU SEE IT?
The edge length s of a cube is an irrational number, the surface area is an irrational number, and the volume is a rational number. Give a possible value of s.



67. MAKE A CONNECTION Write a quadratic function in standard form that has $\sqrt{7}$ and $-\sqrt{7}$ as zeros.

68. MAKING AN ARGUMENT Can you rationalize the denominator of the expression $\frac{2}{\sqrt[3]{5}}$ by multiplying the numerator and denominator by $\sqrt[3]{5}$? Explain.

69. DISCUSS MATHEMATICAL THINKING Determine whether each expression represents a *rational* or an *irrational* number. Explain your reasoning.

a. $4 + \sqrt{6}$

b. $\frac{\sqrt{48}}{\sqrt{3}}$

c. $\frac{8}{\sqrt{12}}$

d. $\sqrt{3} + \sqrt{7}$

e. $\frac{\sqrt[3]{250}}{\sqrt[3]{2}}$

f. $-4\sqrt[3]{12} \cdot 6\sqrt[3]{8}$

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70. THOUGHT PROVOKING Two quantities a and b have a *golden ratio* when the ratio between a and b is the same as the ratio of $a + b$ and a when $a > b$. Use the golden ratio $\dfrac{1+\sqrt{5}}{2}$ and the golden ratio *conjugate* $\dfrac{1-\sqrt{5}}{2}$ for each of the following.

a. Show that the golden ratio and golden ratio conjugate are both solutions of $x^2 - x - 1 = 0$.

b. Construct a geometric diagram that has the golden ratio as the length of a part of the diagram.

71. NUMBER SENSE Fill in the blanks so that the statements are true.

a. $\boxed{}\sqrt{132} = 8\sqrt{\boxed{}}$

b. $5\sqrt[3]{\boxed{}} = 15\sqrt[3]{7}$

c. $-2\sqrt{50} \bullet \boxed{}\sqrt{3} = -30\sqrt{\boxed{}}$

7 MTR **72. INVESTIGATE** Research a profession that uses properties of radicals. Describe how the profession uses radicals to complete a task.

REVIEW & REFRESH

WATCH

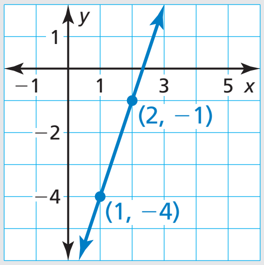
In Exercises 73 and 74, graph the linear equation. Identify the x-intercept.

73. $y = x - 4$

74. $y = -2x + 6$

75. Let $f(x) = x^2 - 12x - 35$ and $g(x) = x - 7$. Find $(fg)(x)$ and $\left(\dfrac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(-3)$ and $\left(\dfrac{f}{g}\right)(8)$.

76. Write an equation of the line that passes through $(3, -2)$ and is (a) parallel and (b) perpendicular to the line shown.



77. Simplify the expression.

$$(x-3)[(x^2 - 5x + 4) - (x^2 - 7x - 1)]$$

In Exercises 78–80, graph the function.

78. $f(x) = (x - 6)^2 + 4$

79. $g(x) = 3x^2 + 6x - 2$

80. $p(x) = -2(x + 1)(x + 5)$

7 MTR **81. MODELING REAL LIFE** The costs for Internet service from two companies are shown. After how many months are the total costs the same at both companies?

	Installation fee	Price per month
Company A	\$60.00	\$42.95
Company B	\$25.00	\$49.95

In Exercises 82–85, simplify the expression.

82. $\sqrt{200}$

83. $\sqrt[3]{\dfrac{54}{343}}$

84. $\dfrac{12}{\sqrt{32}}$

85. $7\sqrt{12} - 4\sqrt{3}$

In Exercises 86–89, solve the equation.

86. $(n + 10)(n - 5) = 0$

87. $b^2 - 12b - 45 = 0$

88. $x^2 + 22x = -121$

89. $4t^2 - 15 = -4t$

90. Write a quadratic function in standard form that models the data.

x	−1	0	1	2	4
y	18	13	10	9	13

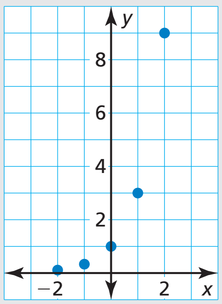
In Exercises 91 and 92, graph the inequality in a coordinate plane.

91. $x \le -5$

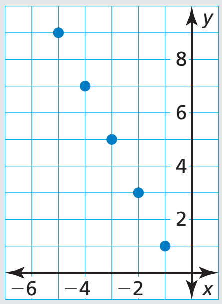
92. $x - 2y < 8$

In Exercises 93 and 94, tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

93.



94.



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