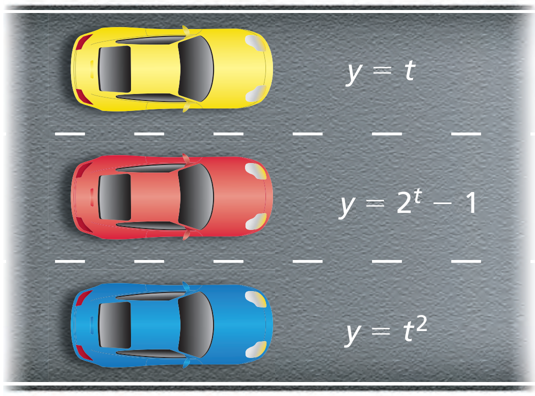
8.7 Combining and Comparing Functions

Learning Target: Combine functions using operations and find average rates of change.

Success Criteria: H
- I can perform operations on functions.
- I can calculate average rates of change.
- I can compare functions using average rates of change.

EXPLORE IT! Comparing Speeds

Work with a partner. Three cars start traveling at the same time. Let y represent the distance (in miles) traveled by each car in t minutes, where $0 \le t \le 5$.



a. Graph the functions in the same coordinate plane for $0 \le t \le 1$. Compare the speeds of the cars. Which car has a constant speed? Which car has the greatest acceleration? Explain your reasoning.

b. Extend the graphs of the functions in the same coordinate plane for $1 \le t \le 5$. Compare the speeds of the cars. How do the values of the three functions compare for greater values of t? Which car has the greatest acceleration? Explain your reasoning.

c. Which of the functions has a growth rate that is eventually much greater than the growth rates of the other functions? Do you think this is true in general? Explain your reasoning.

5 MTR USE STRUCTURE
How does the structure of a function help explain its growth rate?

Functions
MA.912.F.1.3 Calculate and interpret the average rate of change of a real-world situation represented graphically, algebraically, or in a table over a specified interval.
MA.912.F.1.6 Compare key features of linear and nonlinear functions each represented algebraically, graphically, in tables or written descriptions.
H **Also MA.912.F.3.1**

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Operations on Functions

You have learned how to add, subtract, multiply, and divide polynomial expressions. These operations are also defined for functions.

Vocabulary (Az VOCAB)

average rate of change, *p. 447*

KEY IDEA

Operations on Functions

Let f and g be any two functions. A new function can be defined by performing any of the four basic operations on f and g.

Operation	Definition	Example: $f(x) = 2x$, $g(x) = x + 1$
Addition	$(f + g)(x) = f(x) + g(x)$	$(f + g)(x) = 2x + (x + 1) = 3x + 1$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$(f - g)(x) = 2x - (x + 1) = x - 1$
Multiplication	$(fg)(x) = f(x) \cdot g(x)$	$(fg)(x) = 2x(x + 1) = 2x^2 + 2x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x) = \frac{2x}{x + 2}$

The domains of the sum, difference, product, and quotient functions consist of the x-values that are in the domains of both f and g. Additionally, the domain of the quotient does not include x-values for which $g(x) = 0$.

EXAMPLE 1 Performing Operations on Functions

WATCH H

Let $f(x) = 2x^2 - 4x + 5$ and $g(x) = x + 1$.

a. Find $(f + g)(x)$ and state the domain. Then evaluate $(f + g)(2)$.

b. Find $\left(\frac{f}{g}\right)(x)$ and state the domain. Then evaluate $\left(\frac{f}{g}\right)(-3)$.

SOLUTION

a. $(f + g)(x) = f(x) + g(x) = 2x^2 - 4x + 5 + x + 1 = 2x^2 - 3x + 6$

The functions f and g each have the same domain: all real numbers. So, the domain of $f + g$ also consists of all real numbers.

$$(f + g)(2) = 2(2)^2 - 3(2) + 6 = 8 - 6 + 6 = 8$$

b. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2 - 4x + 5}{x + 1}$

The functions f and g each have the same domain: all real numbers. Also, $g(-1) = 0$. So, the domain of $f + g$ is $\{x \mid x \neq -1\}$.

$$\left(\frac{f}{g}\right)(-3) = \frac{2(-3)^2 - 4(-3) + 5}{-3 + 1} = \frac{18 + 12 + 5}{-3 + 1} = \frac{35}{-2} = -17\frac{1}{2}$$

2 MTR USE ANOTHER METHOD

How can you use $f(2)$ and $g(2)$ to find $(f + g)(2)$?

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. Let $f(x) = x^2 - 3x - 2$ and $g(x) = 4x + 3$. Find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $(f + g)(5)$ and $(f - g)(-1)$.

2. Let $f(x) = 7 - 4x$ and $g(x) = x^2 - 1$. Find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(-0.5)$ and $\left(\frac{f}{g}\right)(4)$.

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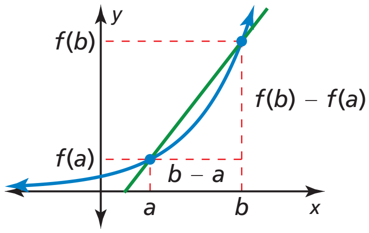
Comparing Functions Using Average Rates of Change

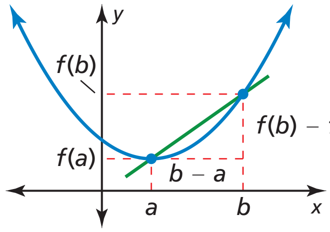
For nonlinear functions, the rate of change is not constant. You can compare two nonlinear functions over the same interval using their *average rates of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

Exponential Function



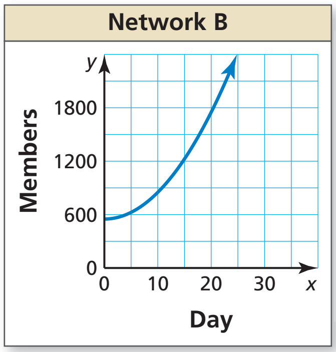
Quadratic Function



EXAMPLE 2 Using and Interpreting Average Rates of Change

Two social media networks open their memberships to the public. Compare the network memberships by calculating and interpreting the average rates of change from Day 10 to Day 20.

Network A	
Day, x	**Members, y**
0	650
5	1025
10	1400
15	1775
20	2150
25	2525



WATCH

SOLUTION

Calculate the average rates of change by using the points whose x-coordinates are 10 and 20.

Network A: Use (10, 1400) and (20, 2150).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{2150 - 1400}{20 - 10} = \frac{750}{10} = 75$$

Network B: Use the graph to estimate the points when $x = 10$ and $x = 20$. Use (10, 850) and (20, 1800).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} \approx \frac{1800 - 850}{20 - 10} = \frac{950}{10} = 95$$

▶ From Day 10 to Day 20, Network A membership increases at an average rate of 75 people per day, and Network B membership increases at an average rate of about 95 people per day. So, Network B membership is growing faster from Day 10 to Day 20.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

3. **VOCABULARY** Describe how to find the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$.

4. Compare the networks in Example 2 by calculating and interpreting the average rates of change from Day 0 to Day 10.

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KEY IDEA

Comparing Functions Using Average Rates of Change

- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing exponential function $y = f(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ or an increasing linear function $y = h(x)$. So, as x increases, $f(x)$ will eventually exceed $g(x)$ or $h(x)$.
- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing linear function $y = h(x)$. So, as x increases, $g(x)$ will eventually exceed $h(x)$.

3 MTR **SELECT METHODS**

How can you use technology to explore these concepts for greater values of x?

7 MTR

EXAMPLE 3 Modeling Real Life

Each year a team of researchers counts the numbers of sea turtle nests on an island and in a coastal wildlife refuge. The function $L(x) = 4x^2 + 72x + 1750$ represents the numbers of sea turtle nests counted on the island x years after 1990. In the wildlife refuge, the team counted 1275 nests in 1990 and observed that the number of nests increased by about 6% each year.

a. From 1990 to 2010, which location had a greater average rate of change?

b. Which location will eventually have a greater number of nests? Explain.

Five species of sea turtles are found in Florida:
- Leatherback
- Loggerhead
- Green Turtle
- Kemp's Ridley
- Hawksbill

SOLUTION

a. Write a function $y = R(x)$ to model the number of nests at the wildlife refuge where x represents the number of years since 1990.

Island: $L(x) = 4x^2 + 72x + 1750$ — Quadratic function

Refuge: $R(x) = 1275(1.06)^x$ — Exponential function

Find the average rate of change from 1990 to 2010 for the number of nests at each location.

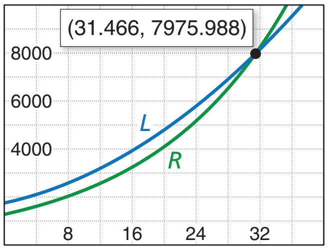
$$\text{Island: } \frac{L(20) - L(0)}{20 - 0} = \frac{4790 - 1750}{20} = \frac{3040}{20} = 152$$

$$\text{Refuge: } \frac{R(20) - R(0)}{20 - 0} \approx \frac{4089 - 1275}{20} = \frac{2814}{20} \approx 141$$

From 1990 to 2010, the average rate of change on the island was greater.

b. The number of nests on the island is represented by a quadratic function, and the number of nests at the refuge is represented by an exponential function. So, the refuge will eventually have a greater number of nests.

Using technology, you can determine that the numbers of nests at each location are about equal when $x \approx 31.5$. So, the number of nests at the refuge exceeds the number of nests on the island between $x = 31$ and $x = 32$, which corresponds to 2021.



SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

5. In Example 2, predict which network will have more members after 50 days. Explain your reasoning.

6. **WHAT IF?** In Example 3, suppose the initial number of nests on the island in 1990 was doubled and maintained the same average rate of change. Does the number of nests at the wildlife refuge still eventually exceed the number of nests on the island? Explain.

GO DIGITAL

8.7 Practice WITH CalcChat® AND CalcView®

In Exercises 1 and 2, find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $f + g$ and $f - g$ for the given value of x. *(See Example 1.)*

1. $f(x) = x^2 + 3x - 18$, $g(x) = -2x + 6$; $x = 3$

2. $f(x) = x^2 - 6x + 8$, $g(x) = x - 5$; $x = -2$

In Exercises 3 and 4, find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate fg and $\frac{f}{g}$ for the given value of x.

3. $f(x) = x^2 + x + 9$, $g(x) = -4x - 3$; $x = -1$

4. $f(x) = -2x^2 - 10x + 5$, $g(x) = -3x + 7$; $x = 3$

5. **MODELING REAL LIFE** The dimensions (in inches) of a rectangular tile can be modeled by $A(x) = x^2 + 5$ and $B(x) = 2x - 1$.

   a. Find $(A + B)(x)$ and $(AB)(x)$. Evaluate and interpret each expression when $x = 4$.

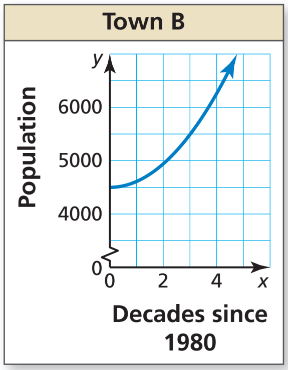
   b. Explain what the expression $(AB)(x)$ represents.

6. **MODELING REAL LIFE** Over an 80-day period, the number of leatherback sea turtle eggs on the two beaches can be modeled by $A(x) = -0.25x^2 + 20x$ and $B(x) = -0.19x^2 + 15.2x$, where x is the number of days.

   a. Find $(A + B)(x)$ and $(A - B)(x)$. Evaluate and interpret each expression when $x = 5$.

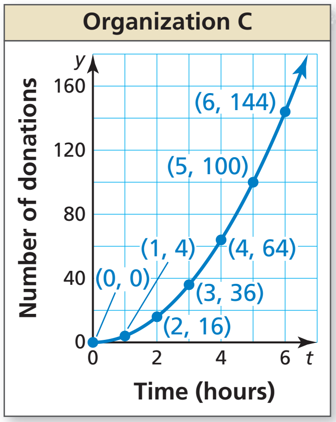
   b. Explain what the expressions in part (a) represent. Find and interpret the vertices, intercepts, and the intervals where $(A + B)(x)$ and $(A - B)(x)$ are increasing and decreasing.

7. **ANALYZING RELATIONSHIPS** The population of Town A in 1980 was 3000. The population of Town A increased by 20% every decade. Let x represent the number of decades since 1980. The graph shows the population of Town B. Compare the populations of the towns by calculating and interpreting the average rates of change from 2000 to 2020. *(See Example 2.)*



8. **ANALYZING RELATIONSHIPS** Three organizations are collecting donations for a cause. Organization A begins with one donation, and the number of donations quadruples each hour. The numbers of donations collected by Organizations B and C are shown.

Organization B	
Time (hours), t	Number of donations, y
0	0
1	4
2	8
3	12
4	16
5	20
6	24



   a. What type of function represents the numbers of donations collected by each organization?

   b. Find the average rates of change of each function for each 1-hour interval from $t = 0$ to $t = 6$.

   c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of donations collected by the three organizations?

9. **MODELING REAL LIFE** Let x represent the number of years since 1970. The function $H(x) = 10x^2 + 10x + 500$ represents the population of Oak Hill. In 1970, Poplar Grove had a population of 200 people. Poplar Grove's population increased by 8% each year. *(See Example 3.)*

   a. From 1970 to 2020, which town's population had a greater average rate of change?

   b. Which town will eventually have a greater population? Explain.

10. **MODELING REAL LIFE** Let x represent the number of years since 2005. The function

$$R(x) = 0.01x^2 + 0.22x + 1.08$$

represents the revenue (in millions of dollars) of Company A. In 2005, Company B had a revenue of \$2.12 million. Company B's revenue increased by \$0.32 million each year.

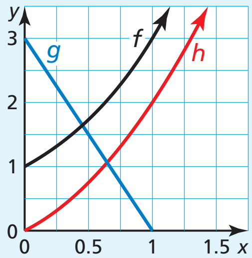
   a. From 2005 to 2020, which company's revenue had a greater average rate of change?

   b. Which company will eventually have a greater revenue? Explain.

GO DIGITAL

11. REASONING Explain why the average rate of change of a linear function is constant and the average rate of change of a quadratic or exponential function is not constant.

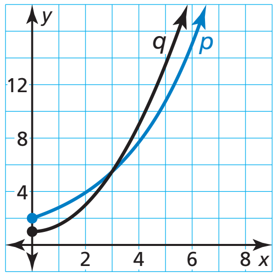
12. HOW DO YOU SEE IT?
Use the graph to order the functions from least to greatest by average rate of change for $0 \le x \le 1$. Explain your reasoning.



13. MODELING REAL LIFE You sell baked goods at a community event. Your revenue can be modeled by $R(x) = 7x^2 - 6x + 2$, and your profit can be modeled by $P(x) = 7x^2 - 11x + 10$, where x is the number of goods sold. Find a function that can be used to model your costs. Justify your answer.

14. THOUGHT PROVOKING
Describe how you can rewrite an absolute value function as two linear functions with domain restrictions. Can this be done with all functions that do not have a constant rate of change? Explain your reasoning.

15. MAKING AN ARGUMENT Function p is an exponential function, and function q is a quadratic function. Your friend says that after about $x = 3$, function q will always have a greater y-value than function p. Is your friend correct? Explain.



REVIEW & REFRESH

In Exercises 16–19, evaluate the expression.

16. $\sqrt{121}$ **17.** $\sqrt[3]{125}$

18. $\sqrt[3]{512}$ **19.** $\sqrt[5]{243}$

In Exercises 20 and 21, plot the points. Tell whether the points can be represented by a *linear*, an *exponential*, or a *quadratic* function.

20.

x	−3	−2	−1	0	1
y	81	27	9	3	1

21.

x	−1	0	1	2	3
y	12	4	−2	−6	−8

In Exercises 22–25, find the product.

22. $(x + 8)(x - 8)$ **23.** $(4y + 2)(4y - 2)$

24. $(3a - 5b)(3a + 5b)$ **25.** $(-2r + 6s)(-2r - 6s)$

In Exercises 26 and 27, using f, graph g. Describe the transformations from the graph of f to the graph g.

26. $f(x) = (x - 4)^2 + 1$; $g(x) = 2f(x)$

27. $f(x) = -(x + 2)(x - 5)$; $g(x) = f(x - 6)$

In Exercises 28–31, solve the equation by graphing. Determine whether the equation has *one solution*, *no solution*, or *infinitely many solutions*.

28. $x - 3 = 5x + 1$ **29.** $3x + 2 = 5x - 2$

30. $2(x - 6) = 2x - 4$ **31.** $4(x + 4) = 2(2x + 8)$

32. Let $f(x) = 3x^2 + 2x - 7$ and $g(x) = x - 9$. Find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(-2)$ and $\left(\frac{f}{g}\right)(3)$.

In Exercises 33–36, tell whether the function has a *maximum value* or a *minimum value*. Then find the value.

33. $y = 2x^2 + 8x + 8$ **34.** $f(x) = 3x^2 - 6x + 1$

35. $f(x) = 4x^2 - 6x + 2$ **36.** $y = -\frac{1}{2}x^2 + 3x - 1$

37. STRUCTURE Without solving completely, determine whether each equation has *no solution*, *one solution*, or *two solutions*. Explain your reasoning.

a. $|x - 2| - 5 = -3$ **b.** $|x + 7| + 6 = 4$