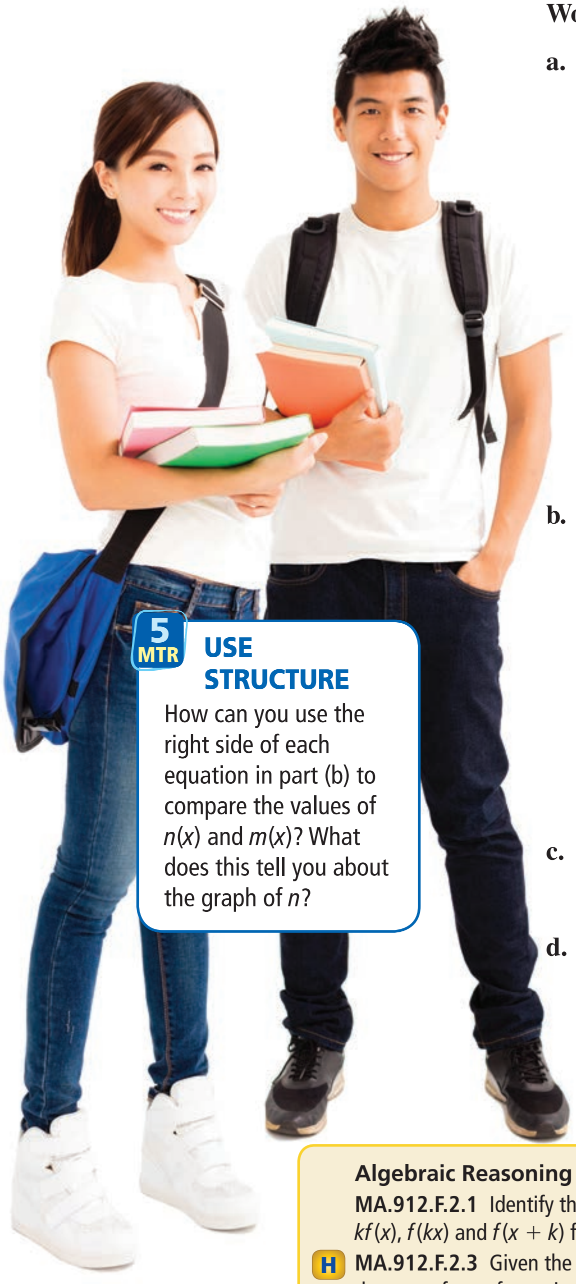
3.7 Transformations of Linear Functions

Learning Target: Graph transformations of linear functions.

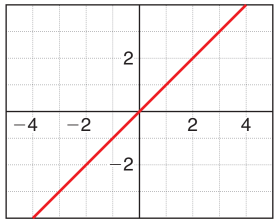
Success Criteria:
- I can identify a transformation of a linear graph.
- I can graph transformations of linear functions.
- I can explain how translations, reflections, stretches, and shrinks affect graphs of functions.

EXPLORE IT! Comparing Graphs of Functions



Work with a partner.

a. The graph of $f(x) = x$ is shown. Graph f and g on the same set of coordinate axes. Compare the graphs of f and g.



i. $g(x) = x + 4$ **ii.** $g(x) = 2x$ **iii.** $g(x) = -x$

b. Write any linear function m in terms of x. Compare the graphs of m and n. Explain your reasoning.

i. $n(x) = m(x) + 3$ **ii.** $n(x) = m(x) - 3$

iii. $n(x) = \frac{1}{3} \cdot m(x)$ **iv.** $n(x) = 3 \cdot m(x)$

v. $n(x) = -m(x)$ **vi.** $n(x) = m(-x)$

c. Discuss your results in part (b) with other students. What do you notice?

d. How does the graph of a function p compare to the graph of each of the following functions? Explain your reasoning.

i. $q(x) = p(x) + k$ **ii.** $q(x) = k \cdot p(x)$, where $k > 0$

iii. $q(x) = -p(x)$ **iv.** $q(x) = p(-x)$

5 MTR USE STRUCTURE
How can you use the right side of each equation in part (b) to compare the values of $n(x)$ and $m(x)$? What does this tell you about the graph of n?

Algebraic Reasoning
MA.912.F.2.1 Identify the effect on the graph or table of a given function after replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$ and $f(x + k)$ for specific values of k.
H **MA.912.F.2.3** Given the graph or table of $f(x)$ and the graph or table of $f(x) + k$, $kf(x)$, $f(kx)$ and $f(x + k)$, state the type of transformation and find the value of the real number k.
Also MA.912.AR.2.4, MA.912.AR.2.5

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Translations and Reflections

A **family of functions** is a group of functions with similar characteristics. The most basic function in a family of functions is the **parent function**. For nonconstant linear functions, the parent function is $f(x) = x$. The graphs of all other nonconstant linear functions are *transformations* of the graph of the parent function. A **transformation** changes the size, shape, position, or orientation of a graph.

WORDS AND MATH

Used as an adjective, *parent* can refer to "being the original source." A *parent function* is the original source of the family of functions.

Vocabulary

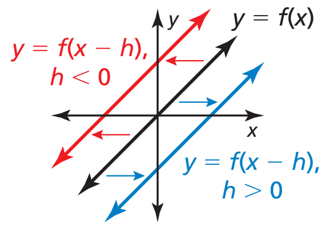
family of functions, *p. 148*
parent function, *p. 148*
transformation, *p. 148*
translation, *p. 148*
reflection, *p. 149*
horizontal shrink, *p. 150*
horizontal stretch, *p. 150*
vertical stretch, *p. 150*
vertical shrink, *p. 150*

KEY IDEAS

A **translation** is a transformation that shifts a graph horizontally or vertically.

Horizontal Translations

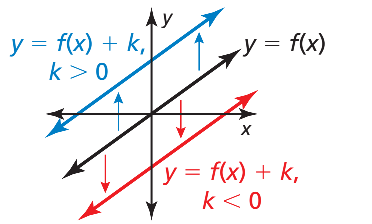
The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$, where $h \neq 0$.



Subtracting h from the *inputs* before evaluating the function shifts the graph left when $h < 0$ and right when $h > 0$.

Vertical Translations

The graph of $y = f(x) + k$ is a vertical translation of the graph of $y = f(x)$, where $k \neq 0$.



Adding k to the *outputs* shifts the graph down when $k < 0$ and up when $k > 0$.

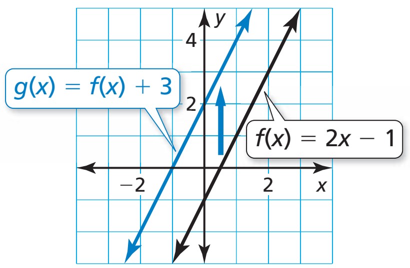
EXAMPLE 1 Describing Horizontal and Vertical Translations

Let $f(x) = 2x - 1$. Graph (a) $g(x) = f(x) + 3$ and (b) $t(x) = f(x + 3)$. Describe the transformations from the graph of f to the graphs of g and t.

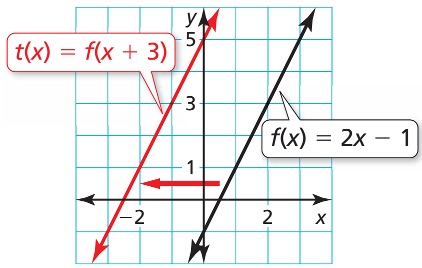
WATCH

SOLUTION

a. The function g is of the form $y = f(x) + k$, where $k = 3$. So, the graph of g is a vertical translation 3 units up of the graph of f.



b. The function t is of the form $y = f(x - h)$, where $h = -3$. So, the graph of t is a horizontal translation 3 units left of the graph of f.



SELF-ASSESSMENT 1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Using f, graph (a) g and (b) h. Describe the transformations from the graph of f to the graphs of g and h.

1. $f(x) = 3x + 1$; $g(x) = f(x) - 2$; $h(x) = f(x - 2)$

2. $f(x) = -2x$; $g(x) = f(x) + 1$; $h(x) = f(x + 1)$

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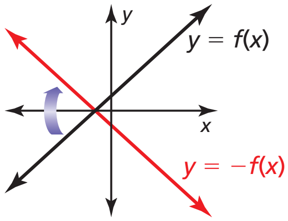
KEY IDEAS

A **reflection** is a transformation that flips a graph over a line called the *line of reflection*.

Reflections in the *x*-Axis

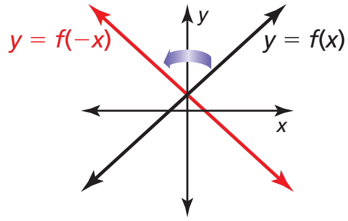
The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.



Multiplying the outputs by -1 changes their signs.

Reflections in the *y*-Axis

The graph of $y = f(-x)$ is a reflection in the y-axis of the graph of $y = f(x)$.



Multiplying the inputs by -1 changes their signs.

STUDY TIP

A reflected point is the same distance from the line of reflection as the original point but on the opposite side of the line.

EXAMPLE 2 Describing Reflections in the *x*-Axis and the *y*-Axis

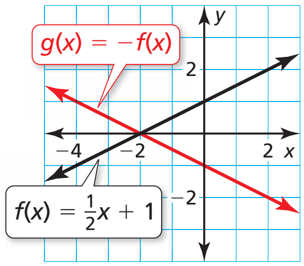
WATCH

Let $f(x) = \frac{1}{2}x + 1$. Graph (a) $g(x) = -f(x)$ and (b) $t(x) = f(-x)$. Describe the transformations from the graph of f to the graphs of g and t.

SOLUTION

a. To find the outputs of g, multiply the outputs of f by -1. The graph of g consists of the points $(x, -f(x))$.

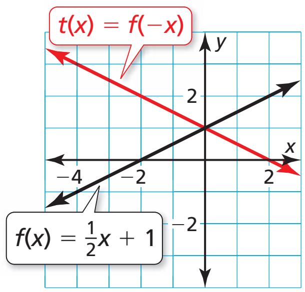
x	-4	-2	0
$f(x)$	-1	0	1
$-f(x)$	1	0	-1



▶ The graph of g is a reflection in the x-axis of the graph of f.

b. To find the outputs of t, multiply the inputs by -1 and then evaluate f. The graph of t consists of the points $(x, f(-x))$.

x	-2	0	2
$-x$	2	0	-2
$f(-x)$	2	1	0



▶ The graph of t is a reflection in the y-axis of the graph of f.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Using f, graph g. Describe the transformation from the graph of f to the graph of g.

3. $f(x) = \frac{3}{2}x + 2$; $g(x) = -f(x)$

4. $f(x) = -4x - 2$; $g(x) = f(-x)$

5. **OPEN-ENDED** Write a linear function for which a reflection in the x-axis has the same graph as a reflection in the y-axis.

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Stretches and Shrinks

You can transform a function by multiplying all the inputs (x-coordinates) by the same factor a. When $a > 1$, the transformation is a **horizontal shrink** because the graph shrinks toward the y-axis. When $0 < a < 1$, the transformation is a **horizontal stretch** because the graph stretches away from the y-axis. In each case, the y-intercept stays the same.

You can also transform a function by multiplying all the outputs (y-coordinates) by the same factor a. When $a > 1$, the transformation is a **vertical stretch** because the graph stretches away from the x-axis. When $0 < a < 1$, the transformation is a **vertical shrink** because the graph shrinks toward the x-axis. In each case, the x-intercept stays the same.

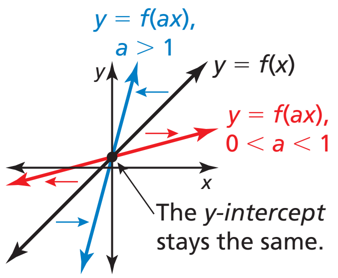
STUDY TIP

When $a < 0$ and $a \neq -1$, the graphs of $y = f(ax)$ and $y = a \cdot f(x)$ represent a stretch or shrink *and* a reflection in the x- or y-axis of the graph of $y = f(x)$.

KEY IDEAS

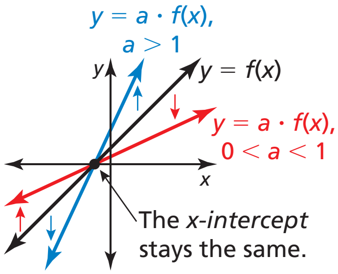
Horizontal Stretches and Shrinks

The graph of $y = f(ax)$ is a horizontal stretch or shrink by a factor of $\frac{1}{a}$ of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.



Vertical Stretches and Shrinks

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.



EXAMPLE 3 Describing Horizontal and Vertical Stretches

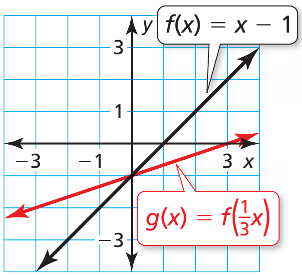
WATCH

Let $f(x) = x - 1$. Graph (a) $g(x) = f\left(\frac{1}{3}x\right)$ and (b) $h(x) = 3f(x)$. Describe the transformations from the graph of f to the graphs of g and h.

SOLUTION

a. To find the outputs of g, multiply the inputs by $\frac{1}{3}$. Then evaluate f. The graph of g consists of the points $\left(x, f\left(\frac{1}{3}x\right)\right)$.

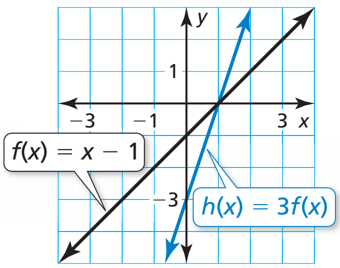
x	-3	0	3
$\frac{1}{3}x$	-1	0	1
$f\left(\frac{1}{3}x\right)$	-2	-1	0



▶ The graph of g is a horizontal stretch of the graph of f by a factor of $1 \div \frac{1}{3} = 3$.

b. To find the outputs of h, multiply the outputs of f by 3. The graph of h consists of the points $(x, 3f(x))$.

x	0	1	2
$f(x)$	-1	0	1
$3f(x)$	-3	0	3



▶ The graph of h is a vertical stretch of the graph of f by a factor of 3.

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EXAMPLE 4 Describing Horizontal and Vertical Shrinks

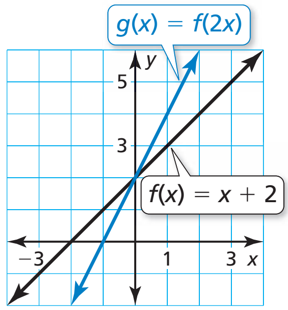
WATCH

Let $f(x) = x + 2$. Graph (a) $g(x) = f(2x)$ and (b) $h(x) = \frac{1}{4}f(x)$. Describe the transformations from the graph of f to the graphs of g and h.

SOLUTION

a. To find the outputs of g, multiply the inputs by 2. Then evaluate f. The graph of g consists of the points $(x, f(2x))$.

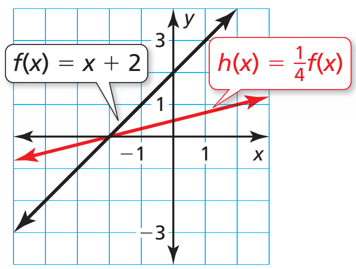
x	-1	0	1
$2x$	-2	0	2
$f(2x)$	0	2	4



▶ The graph of g is a horizontal shrink of the graph of f by a factor of $\frac{1}{2}$.

b. To find the outputs of h, multiply the outputs of f by $\frac{1}{4}$. The graph of h consists of the points $\left(x, \frac{1}{4}f(x)\right)$.

x	-2	0	2
$f(x)$	0	2	4
$\frac{1}{4}f(x)$	0	$\frac{1}{2}$	1



▶ The graph of h is a vertical shrink of the graph of f by a factor of $\frac{1}{4}$.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

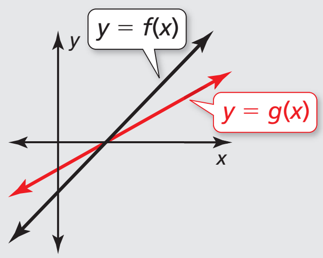
Using f, graph (a) g and (b) h. Describe the transformations from the graph of f to the graphs of g and h.

6. $f(x) = 4x - 2$; $g(x) = f\left(\frac{1}{2}x\right)$; $h(x) = 2f(x)$

7. $f(x) = -3x + 4$; $g(x) = f(3x)$; $h(x) = \frac{1}{2}f(x)$

8. WRITING How does the value of a in the equation $y = h(ax)$ affect the graph of $y = h(x)$? How does the value of a in the equation $y = a \cdot h(x)$ affect the graph of $y = h(x)$?

9. REASONING The functions f and g are linear functions. The graph of g is a vertical shrink of the graph of f. What can you say about the intercepts of the graphs of f and g? Is this always true? Explain.



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Describing Transformations

EXAMPLE 5 Describing a Transformation

The table represents two linear functions *f* and *g*. Describe the transformation from the graph of *f* to the graph of *g*.

x	-1	0	1	2	3
$f(x)$	-4	-1	2	5	8
$g(x)$	-2	1	4	7	10

SOLUTION

To determine the transformation, compare values of $f(x)$ and $g(x)$.

x	-1	0	1	2	3
$f(x)$	-4	-1	2	5	8
	$+2$	$+2$	$+2$	$+2$	$+2$
$g(x)$	-2	1	4	7	10

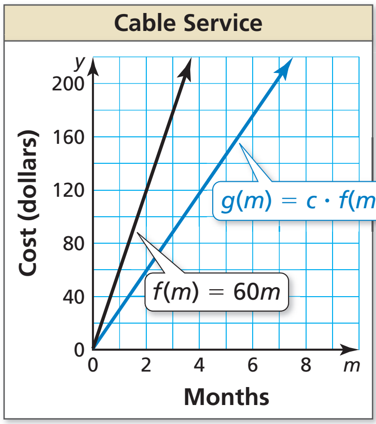
For each input, $g(x)$ is 2 more than $f(x)$. So, $g(x) = f(x) + 2$.

▶ The graph of *g* is a translation 2 units up of the graph of *f*.

EXAMPLE 6 Modeling Real Life

7 MTR

The cost (in dollars) of cable service for *m* months is represented by the function *f*. To attract new customers, the cable company multiplies the monthly fee by a factor of *c*. Use the graph to find and interpret the value of *c*.



SOLUTION

To find *c*, compare *f* and *g*. The function $g(m) = c \cdot f(m)$ indicates that the graph of *g* is a vertical stretch or shrink of the graph of *f*. The graphs of *f* and *g* show that for any number of months, the new cost is one-half of the original cost. For example, the cost for two months ($m = 2$) decreases from \$120 to \$60. So, $c = \frac{1}{2}$.

▶ The factor $c = \frac{1}{2}$ indicates that the monthly price is halved.

SELF-ASSESSMENT

1 I don't understand yet. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

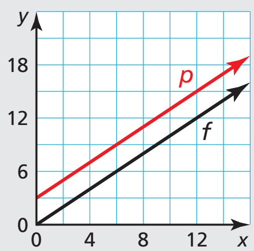
10. The table represents two linear functions *f* and *g*. Describe the transformation from the graph of *f* to the graph of *g*.

x	0	2	4	6	8
$f(x)$	-8	-4	0	4	8
$g(x)$	-2	-1	0	1	2

11. A company pays *x* dollars per unit for a product. The selling price is represented by the function *p*.

a. What does $f(x) = x$ represent in this situation? Describe a transformation of the graph of *f* that results in the graph of *p*.

b. How does the company determine the selling price of a product?



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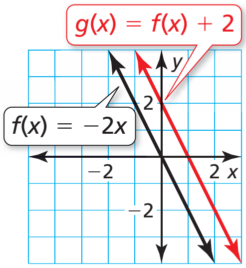
3.7 Practice WITH CalcChat® AND CalcView®

In Exercises 1–6, use the graphs of *f* and *g* to describe the transformation from the graph of *f* to the graph of *g*. *(See Example 1.)*

1.



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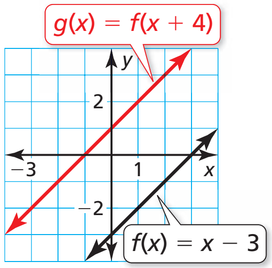


3. $f(x) = \frac{1}{3}x + 3$; $g(x) = f(x) - 3$

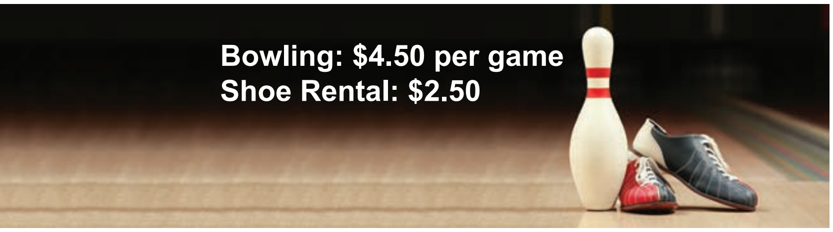
4. $f(x) = -3x + 4$; $g(x) = f(x) + 1$

5. $f(x) = -x - 2$; $g(x) = f(x + 5)$

6. $f(x) = \frac{1}{2}x - 5$; $g(x) = f(x - 3)$

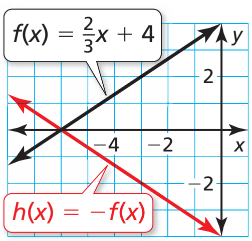
7. MODELING REAL LIFE You and your friend start biking from the same location. Your distance (in miles) after *t* minutes is represented by $d(t) = \frac{1}{5}t$. Your friend starts biking 5 minutes after you. Her distance is represented by $f(t) = d(t - 5)$. Describe the transformation from the graph of *d* to the graph of *f*.

8. MODELING REAL LIFE The total cost (in dollars) to bowl *n* games is represented by $C(n) = 4.5n + 2.5$. The shoe rental price increases \$0.50. The new total cost is represented by $T(n) = C(n) + 0.5$. Describe the transformation from the graph of *C* to the graph of *T*.



In Exercises 9–12, use the graphs of *f* and *h* to describe the transformation from the graph of *f* to the graph of *h*. *(See Example 2.)*

9.



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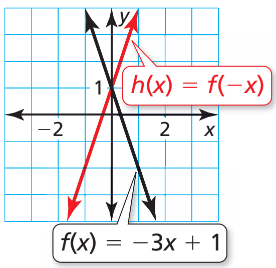
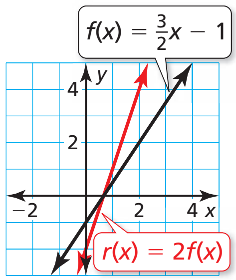


11. $f(x) = -5 - x$; $h(x) = f(-x)$

12. $f(x) = \frac{1}{4}x - 2$; $h(x) = -f(x)$

In Exercises 13–18, use the graphs of *f* and *r* to describe the transformation from the graph of *f* to the graph of *r*. *(See Example 3.)*

13.



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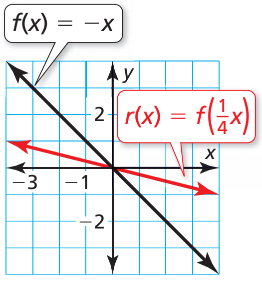


15. $f(x) = -2x - 4$; $r(x) = f\left(\frac{1}{2}x\right)$

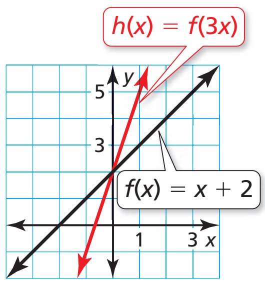
16. $f(x) = 3x + 5$; $r(x) = f\left(\frac{1}{3}x\right)$

17. $f(x) = \frac{2}{3}x + 1$; $r(x) = 3f(x)$

18. $f(x) = -\frac{1}{4}x - 2$; $r(x) = 4f(x)$

In Exercises 19–24, use the graphs of *f* and *h* to describe the transformation from the graph of *f* to the graph of *h*. *(See Example 4.)*

19.



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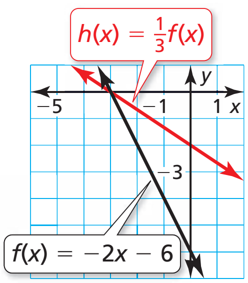

21. $f(x) = 3x - 12$; $h(x) = \frac{1}{6}f(x)$

22. $f(x) = -x + 1$; $h(x) = f(2x)$

23. $f(x) = -2x - 2$; $h(x) = f(5x)$

24. $f(x) = 4x + 8$; $h(x) = \frac{3}{4}f(x)$

25. MODELING REAL LIFE The temperature (in degrees Fahrenheit) *x* hours after 5 P.M. is represented by $t(x) = -4x + 72$. The temperature *x* hours after 10 A.M. is represented by $d(x) = 4x + 72$. Describe the transformation from the graph of *t* to the graph of *d*.

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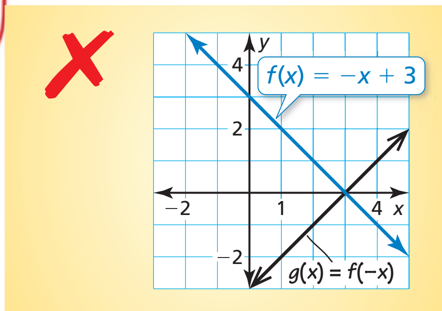
38.



26. **MODELING REAL LIFE** The cost (in dollars) of a basic music streaming service for m months is represented by $B(m) = 5m$. The cost of the premium service is represented by $P(m) = 10m$. Describe the transformation from the graph of B to the graph of P.

In Exercises 27–32, use the graphs of f and g to describe the transformation from the graph of f to the graph of g.

27. $f(x) = x - 2$; $g(x) = f(x + 4)$

28. $f(x) = -4x + 8$; $g(x) = -f(x)$

29. $f(x) = -2x - 7$; $g(x) = f(x - 2)$

30. $f(x) = 3x + 8$; $g(x) = f\left(\frac{2}{3}x\right)$

31. $f(x) = x - 6$; $g(x) = 6f(x)$

32. $f(x) = -x$; $g(x) = f(x) - 3$

In Exercises 33–36, write a function g in terms of f so that the statement is true.

33. The graph of g is a horizontal translation 2 units right of the graph of f.

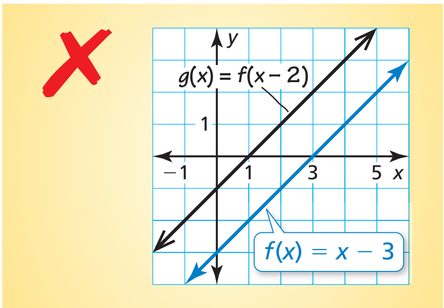
34. The graph of g is a reflection in the y-axis of the graph of f.

35. The graph of g is a vertical translation 4 units up of the graph of f.

36. The graph of g is a horizontal shrink by a factor of $\frac{1}{5}$ of the graph of f.

ERROR ANALYSIS **In Exercises 37 and 38, describe and correct the error in graphing g.**

37.



In Exercises 39–46, describe the transformation from the graph of f to the graph of g. *(See Example 5.)*

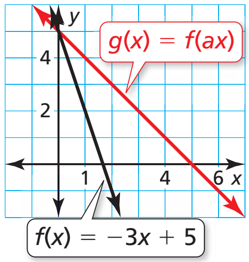
39.

x	−2	−1	0	1	2
$f(x)$	14	11	8	5	2
$g(x) = f(x + k)$	11	8	5	2	−1

40.

x	−3	−2	−1	0	1
$f(x)$	−10	−6	−2	2	6
$g(x) = f(x) + k$	−5	−1	3	7	11

41.



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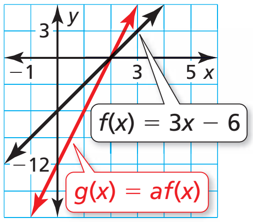


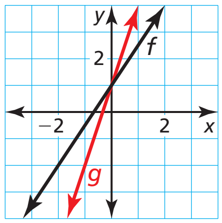
43.

x	−1	0	1	2	3
$f(x)$	6	7	8	9	10
$g(x)$	−6	−7	−8	−9	−10

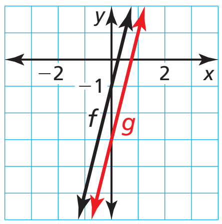
44.

x	3	5	7	9	11
$f(x)$	2	0	−2	−4	−6
$g(x)$	−2	−4	−6	−8	−10

45.



46.



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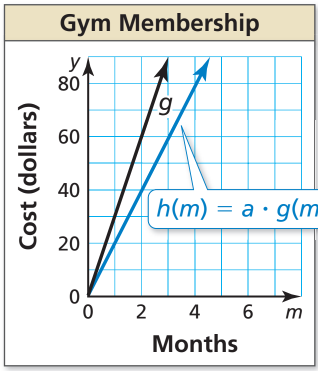
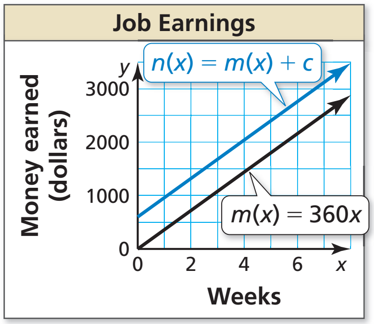
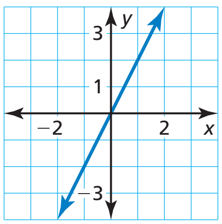
47. MODELING REAL LIFE The cost (in dollars) of a gym membership for m months is represented by the function g. In January, the gym multiplies the monthly fee by a factor of a. Use the graph to find and interpret the value of a. *(See Example 6.)*



48. MODELING REAL LIFE The function $m(x) = 360x$ represents the amount of money (in dollars) you make at your job after x weeks. You earn a bonus of c dollars. Use the graph to find and interpret the value of c.



49. B.E.S.T. TEST PREP Which of the graphs are related by only a translation? Explain.

Ⓐ


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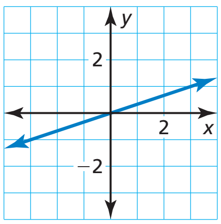

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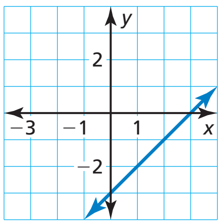

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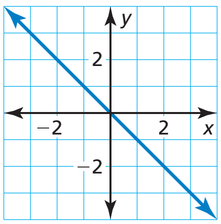

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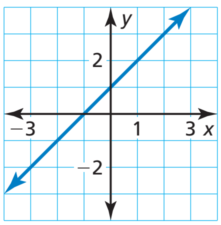

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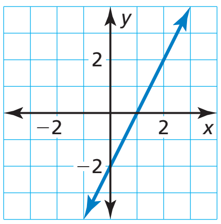

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50. WRITING How does the value of p in the equation $y = g(x) + p$ affect the graph of $y = g(x)$? How does the value of p in the equation $y = g(x + p)$ affect the graph of $y = g(x)$?

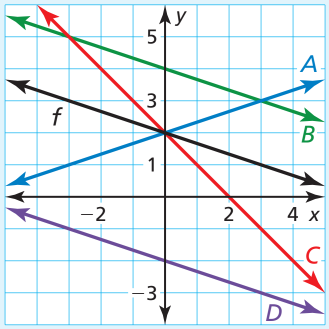
51. STRUCTURE The graph of $g(x) = a \cdot f(x - b) + c$ is a transformation of the graph of the linear function f. Complete each statement.

a. The graph of g is a vertical ______ of the graph of f when $a = 4$, $b = 0$, and $c = 0$.

b. The graph of g is a vertical translation 1 unit up of the graph of f when $a = 1$, $b = 0$, and $c =$ _____.

c. The graph of g is a reflection in the ______ of the graph of f when $a = -1$, $b = 0$, and $c = 0$.

52. HOW DO YOU SEE IT?
Match each function with its graph. Explain.



a. $a(x) = f(-x)$ **b.** $g(x) = f(x) - 4$

c. $h(x) = f(x) + 2$ **d.** $k(x) = f(3x)$

OPEN-ENDED In Exercises 53 and 54, write a function whose graph passes through the given point and is a transformation of the graph of $f(x) = x$.

53. $(4, 2)$ **54.** $\left(\frac{3}{2}, \frac{7}{2}\right)$

In Exercises 55–58, graph f and g. Write g in terms of f. Describe the transformation from the graph of f to the graph of g.

55. $f(x) = 2x - 5$; $g(x) = 2x - 8$

56. $f(x) = 3x + 9$; $g(x) = 3x + 15$

57. $f(x) = -x - 4$; $g(x) = x - 4$

58. $f(x) = x - 1$; $g(x) = 3x - 3$

59. REASONING The graph of $f(x) = x + 5$ is a vertical translation 5 units up of the graph of $f(x) = x$. How can you obtain the graph of $f(x) = x + 5$ from the graph of $f(x) = x$ using a horizontal translation?

60. REASONING A swimming pool is filled with water by a hose at a rate of 1020 gallons per hour. The amount (in gallons) of water in the pool after t hours is represented by the function $v(t) = 1020t$. How does the graph of v change in each situation?

a. A larger hose is found. Then the pool is filled at a rate of 1360 gallons per hour.

b. Before filling up the pool with a hose, a water truck adds 2000 gallons of water to the pool.

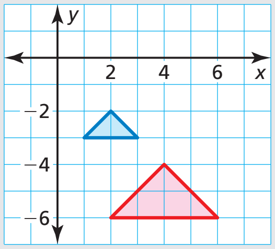
4 MTR **61. MAKING AN ARGUMENT** Is it true that for all linear functions, a horizontal stretch by a factor of c produces the same result as a vertical shrink by a factor of $\frac{1}{c}$? Explain.

62. THOUGHT PROVOKING
When is the graph of $y = f(x) + w$ the same as the graph of $y = f(x + w)$ for linear functions? Explain your reasoning.

REVIEW & REFRESH

WATCH

63. The red figure is similar to the blue figure. Describe a similarity transformation between the figures.



In Exercises 64 and 65, solve the inequality. Graph the solution, if possible.

64. $5|x + 7| < 25$

65. $-2|x + 1| \geq 18$

66. REASONING Complete the inequality $-\frac{1}{6}n$ ___ $\frac{2}{3}$ with <, ≤, >, or ≥ so that the solution is $n \leq -4$.

67. Evaluate $g(x) = \frac{1}{4}x - 5$ when $x = 12$ and when $x = -2$.

7 MTR **68. MODELING REAL LIFE** An elevator on the top floor of a building begins to descend to the ground floor. The function $h(t) = -8t + 250$ models the situation, where $h(t)$ is the height (in meters) of the elevator t seconds after it begins to descend.

a. Graph the function, and find its domain and range.

b. Interpret the terms and coefficient in the equation, and the x-intercept of its graph.

In Exercises 69–72, solve the equation. Check your solution.

69. $2.5b = 10$

70. $\frac{c}{5} + 1 = -2$

71. $14 - 3q = 4q - 14 + q$

72. $|-4r + 6| - 5 = 13$

7 MTR **73. MODELING REAL LIFE** The linear function $m = 55 - 8.5b$ represents the amount m (in dollars) of money that you have after buying b books.

a. Find the domain of the function. Is the domain discrete or continuous? Explain.

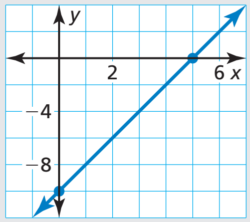
b. Graph the function using its domain.

In Exercises 74 and 75, graph f and h. Describe the transformation from the graph of f to the graph of h.

74. $f(x) = x$; $h(x) = \frac{1}{3}x$

75. $f(x) = x$; $h(x) = x - 4$

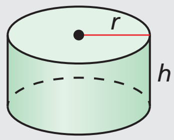
5 MTR **76. STRUCTURE** The graph of the equation $Ax + By = 15$ is shown. Find the values of A and B.



77. Determine whether the relation is a function. Explain.

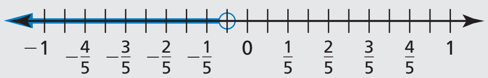
(−10, 2), (−8, 3), (−6, 5), (−8, 8), (−10, 6)

78. Solve the formula for h.



$V = \pi r^2 h$

79. Write an inequality that represents the graph.



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