9.1 Right Triangle Trigonometry

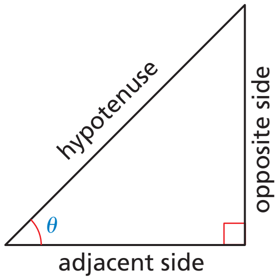
Essential Question

How can you find a trigonometric function of an acute angle θ?

FLORIDA STANDARDS

Preparing for
MAFS.912.F-TF.1.1
MAFS.912.F-TF.1.2
MAFS.912.F-TF.2.5
MAFS.912.F-TF.3.8

Consider one of the acute angles θ of a right triangle. Ratios of a right triangle's side lengths are used to define the six *trigonometric functions*, as shown.



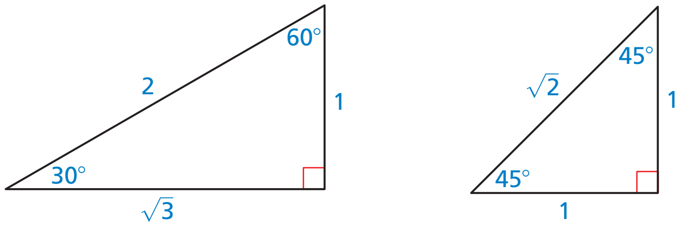
Sine $\sin \theta = \dfrac{\text{opp.}}{\text{hyp.}}$ **Cosine** $\cos \theta = \dfrac{\text{adj.}}{\text{hyp.}}$

Tangent $\tan \theta = \dfrac{\text{opp.}}{\text{adj.}}$ **Cotangent** $\cot \theta = \dfrac{\text{adj.}}{\text{opp.}}$

Secant $\sec \theta = \dfrac{\text{hyp.}}{\text{adj.}}$ **Cosecant** $\csc \theta = \dfrac{\text{hyp.}}{\text{opp.}}$

EXPLORATION 1 Trigonometric Functions of Special Angles

Work with a partner. Find the exact values of the sine, cosine, and tangent functions for the angles 30°, 45°, and 60° in the right triangles shown.



CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results in constructing arguments.

EXPLORATION 2 Exploring Trigonometric Identities

Work with a partner.

Use the definitions of the trigonometric functions to explain why each *trigonometric identity* is true.

a. $\sin \theta = \cos(90° - \theta)$

b. $\cos \theta = \sin(90° - \theta)$

c. $\sin \theta = \dfrac{1}{\csc \theta}$

d. $\tan \theta = \dfrac{1}{\cot \theta}$

Use the definitions of the trigonometric functions to complete each trigonometric identity.

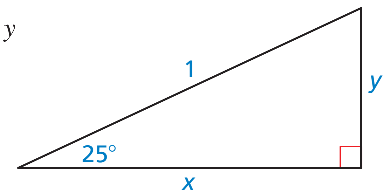
e. $(\sin \theta)^2 + (\cos \theta)^2 =$ ______

f. $(\sec \theta)^2 - (\tan \theta)^2 =$ ______

Communicate Your Answer

3. How can you find a trigonometric function of an acute angle θ?

4. Use a calculator to find the lengths x and y of the legs of the right triangle shown.



9.1 Lesson

Core Vocabulary

sine, *p. 462*
cosine, *p. 462*
tangent, *p. 462*
cosecant, *p. 462*
secant, *p. 462*
cotangent, *p. 462*

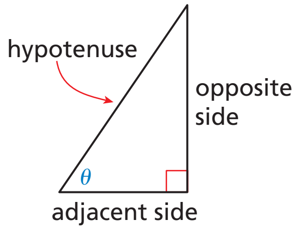
Previous
right triangle
hypotenuse
acute angle
Pythagorean Theorem
reciprocal
complementary angles

What You Will Learn

- Evaluate trigonometric functions of acute angles.
- Find unknown side lengths and angle measures of right triangles.
- Use trigonometric functions to solve real-life problems.

The Six Trigonometric Functions

Consider a right triangle that has an acute angle θ (the Greek letter *theta*). The three sides of the triangle are the *hypotenuse*, the side *opposite* θ, and the side *adjacent* to θ.



Ratios of a right triangle's side lengths are used to define the six trigonometric functions: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, and **cotangent**. These six functions are abbreviated sin, cos, tan, csc, sec, and cot, respectively.

Core Concept

Right Triangle Definitions of Trigonometric Functions

Let θ be an acute angle of a right triangle. The six trigonometric functions of θ are defined as shown.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

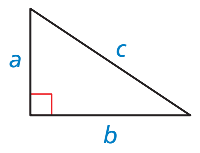
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

The abbreviations *opp.*, *adj.*, and *hyp.* are often used to represent the side lengths of the right triangle. Note that the ratios in the second row are reciprocals of the ratios in the first row.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

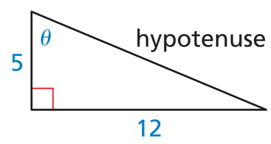
REMEMBER

The Pythagorean Theorem states that $a^2 + b^2 = c^2$ for a right triangle with hypotenuse of length c and legs of lengths a and b.



EXAMPLE 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions of the angle θ.



SOLUTION

From the Pythagorean Theorem, the length of the hypotenuse is

$$\begin{aligned}\text{hyp.} &= \sqrt{5^2 + 12^2}\\ &= \sqrt{169}\\ &= 13.\end{aligned}$$

Using adj. $= 5$, opp. $= 12$, and hyp. $= 13$, the values of the six trigonometric functions of θ are:

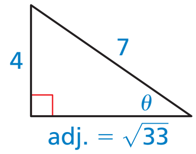
$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{12}{13} \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{5}{13} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{12}{5}$$

$$\csc\theta = \frac{\text{hyp.}}{\text{opp.}} = \frac{13}{12} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{13}{5} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{5}{12}$$

EXAMPLE 2 Evaluating Trigonometric Functions

In a right triangle, θ is an acute angle and $\sin \theta = \frac{4}{7}$. Evaluate the other five trigonometric functions of θ.

SOLUTION



Step 1 Draw a right triangle with acute angle θ such that the leg opposite θ has length 4 and the hypotenuse has length 7.

Step 2 Find the length of the adjacent side. By the Pythagorean Theorem, the length of the other leg is

$$\text{adj.} = \sqrt{7^2 - 4^2} = \sqrt{33}.$$

Step 3 Find the values of the remaining five trigonometric functions.

Because $\sin \theta = \frac{4}{7}$, $\csc \theta = \frac{\text{hyp.}}{\text{opp.}} = \frac{7}{4}$. The other values are:

$$\cos \theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{33}}{7} \qquad \tan \theta = \frac{\text{opp.}}{\text{adj.}} = \frac{4}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$$

$$\sec \theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33} \qquad \cot \theta = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{33}}{4}$$

Monitoring Progress

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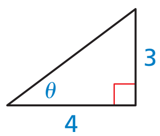
Evaluate the six trigonometric functions of the angle θ.

1.



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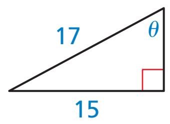

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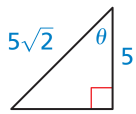


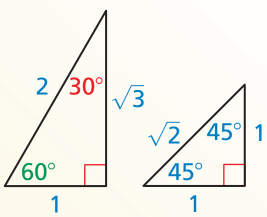
4. In a right triangle, θ is an acute angle and $\cos \theta = \frac{7}{10}$. Evaluate the other five trigonometric functions of θ.

The angles 30°, 45°, and 60° occur frequently in trigonometry. You can use the trigonometric values for these angles to find unknown side lengths in special right triangles.

Core Concept

Trigonometric Values for Special Angles

The table gives the values of the six trigonometric functions for the angles 30°, 45°, and 60°. You can obtain these values from the triangles shown.

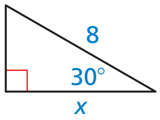


θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Finding Side Lengths and Angle Measures

EXAMPLE 3 Finding an Unknown Side Length

Find the value of x for the right triangle.



SOLUTION

Write an equation using a trigonometric function that involves the ratio of x and 8. Solve the equation for x.

$\cos 30° = \frac{\text{adj.}}{\text{hyp.}}$ — Write trigonometric equation.

$\frac{\sqrt{3}}{2} = \frac{x}{8}$ — Substitute.

$4\sqrt{3} = x$ — Multiply each side by 8.

▶ The length of the side is $x = 4\sqrt{3} \approx 6.93$.

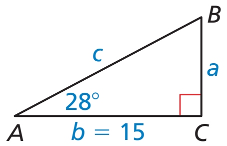
READING

Throughout this chapter, a capital letter is used to denote both an angle of a triangle and its measure. The same letter in lowercase is used to denote the length of the side opposite that angle.

Finding all unknown side lengths and angle measures of a triangle is called *solving the triangle*. Solving right triangles that have acute angles other than 30°, 45°, and 60° may require the use of a calculator. Be sure the calculator is set in *degree* mode.

EXAMPLE 4 Using a Calculator to Solve a Right Triangle

Solve $\triangle ABC$.



SOLUTION

Because the triangle is a right triangle, A and B are complementary angles. So, $B = 90° - 28° = 62°$.

Next, write two equations using trigonometric functions, one that involves the ratio of a and 15, and one that involves c and 15. Solve the first equation for a and the second equation for c.

$\tan 28° = \frac{\text{opp.}}{\text{adj.}}$	Write trigonometric equation.	$\sec 28° = \frac{\text{hyp.}}{\text{adj.}}$
$\tan 28° = \frac{a}{15}$	Substitute.	$\sec 28° = \frac{c}{15}$
$15(\tan 28°) = a$	Solve for the variable.	$15\left(\frac{1}{\cos 28°}\right) = c$
$7.98 \approx a$	Use a calculator.	$16.99 \approx c$

▶ So, $B = 62°$, $a \approx 7.98$, and $c \approx 16.99$.

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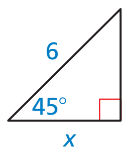
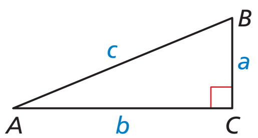
5. Find the value of x for the right triangle shown.





Solve $\triangle ABC$ using the diagram at the left and the given measurements.

6. $B = 45°, c = 5$

7. $A = 32°, b = 10$

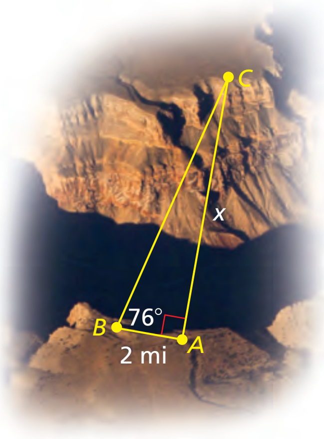
8. $A = 71°, c = 20$

9. $B = 60°, a = 7$

Solving Real-Life Problems

EXAMPLE 5 **Using Indirect Measurement**

You are hiking near a canyon. While standing at A, you measure an angle of 90º between B and C, as shown. You then walk to B and measure an angle of 76° between A and C. The distance between A and B is about 2 miles. How wide is the canyon between A and C?



FINDING AN ENTRY POINT

The tangent function is used to find the unknown distance because it involves the ratio of x and 2.

SOLUTION

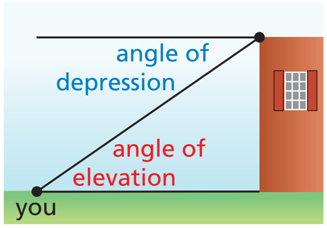
$\tan 76° = \frac{x}{2}$ — Write trigonometric equation.

$2(\tan 76°) = x$ — Multiply each side by 2.

$8.0 \approx x$ — Use a calculator.

▶ The width is about 8.0 miles.

If you look at a point above you, such as the top of a building, the angle that your line of sight makes with a line parallel to the ground is called the *angle of elevation*. At the top of the building, the angle between a line parallel to the ground and your line of sight is called the *angle of depression*. These two angles have the same measure.

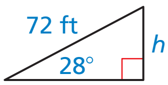


EXAMPLE 6 **Using an Angle of Elevation**

A parasailer is attached to a boat with a rope 72 feet long. The angle of elevation from the boat to the parasailer is 28°. Estimate the parasailer's height above the boat.

SOLUTION

Step 1 Draw a diagram that represents the situation.



Step 2 Write and solve an equation to find the height h.

$\sin 28° = \frac{h}{72}$ — Write trigonometric equation.

$72(\sin 28°) = h$ — Multiply each side by 72.

$33.8 \approx h$ — Use a calculator.

▶ The height of the parasailer above the boat is about 33.8 feet.

Monitoring Progress

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10. In Example 5, find the distance between B and C.

11. **WHAT IF?** In Example 6, estimate the height of the parasailer above the boat when the angle of elevation is 38°.

9.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE In a right triangle, the two trigonometric functions of θ that are defined using the lengths of the hypotenuse and the side adjacent to θ are ___________ and ___________.

2. VOCABULARY Compare an angle of elevation to an angle of depression.

3. WRITING Explain what it means to solve a right triangle.

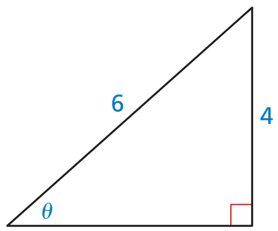
4. DIFFERENT WORDS, SAME QUESTION Which is different? Find "both" answers.

What is the cosecant of θ?

What is $\frac{1}{\sin\ \theta}$?

What is the ratio of the side opposite θ to the hypotenuse?

What is the ratio of the hypotenuse to the side opposite θ?



Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, evaluate the six trigonometric functions of the angle θ. *(See Example 1.)*

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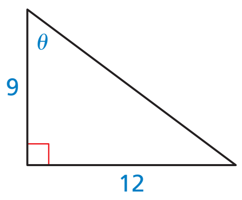

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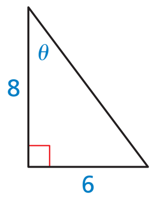

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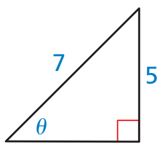

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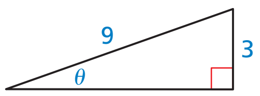

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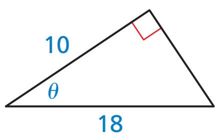

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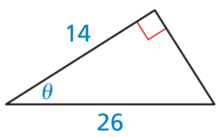


11. REASONING Let θ be an acute angle of a right triangle. Use the two trigonometric functions $\tan\ \theta = \frac{4}{9}$ and $\sec\ \theta = \frac{\sqrt{97}}{9}$ to sketch and label the right triangle. Then evaluate the other four trigonometric functions of θ.

12. ANALYZING RELATIONSHIPS Evaluate the six trigonometric functions of the $90° - \theta$ angle in Exercises 5–10. Describe the relationships you notice.

In Exercises 13–18, let θ be an acute angle of a right triangle. Evaluate the other five trigonometric functions of θ. *(See Example 2.)*

13. $\sin\ \theta = \frac{7}{11}$

14. $\cos\ \theta = \frac{5}{12}$

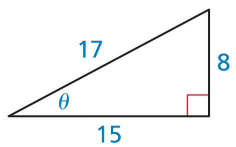
15. $\tan\ \theta = \frac{7}{6}$

16. $\csc\ \theta = \frac{15}{8}$

17. $\sec\ \theta = \frac{14}{9}$

18. $\cot\ \theta = \frac{16}{11}$

19. ERROR ANALYSIS Describe and correct the error in finding $\sin\ \theta$ of the triangle below.

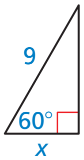


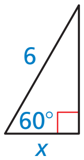
$$\sin\ \theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{15}{17}$$

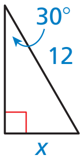
20. **ERROR ANALYSIS** Describe and correct the error in finding csc θ, given that θ is an acute angle of a right triangle and $\cos\theta = \frac{7}{11}$.

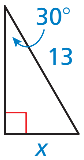
✗ $\csc\theta = \dfrac{1}{\cos\theta} = \dfrac{11}{7}$

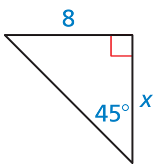
In Exercises 21–26, find the value of x for the right triangle. *(See Example 3.)*

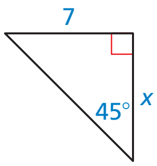
21. 

22. 

23. 

24. 

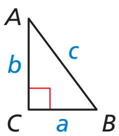
25. 

26. 

USING TOOLS **In Exercises 27–32, evaluate the trigonometric function using a calculator. Round your answer to four decimal places.**

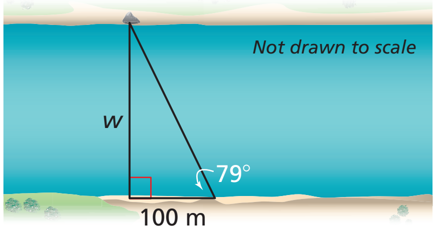
27. cos 14°
28. tan 31°
29. csc 59°
30. sin 23°
31. cot 6°
32. sec 11°

In Exercises 33–40, solve $\triangle ABC$ using the diagram and the given measurements. *(See Example 4.)*



33. $B = 36°$, $a = 23$
34. $A = 27°$, $b = 9$
35. $A = 55°$, $a = 17$
36. $B = 16°$, $b = 14$
37. $A = 43°$, $b = 31$
38. $B = 31°$, $a = 23$
39. $B = 72°$, $c = 12.8$
40. $A = 64°$, $a = 7.4$

41. **MODELING WITH MATHEMATICS** To measure the width of a river, you plant a stake on one side of the river, directly across from a boulder. You then walk 100 meters to the right of the stake and measure a 79° angle between the stake and the boulder. What is the width w of the river? *(See Example 5.)*

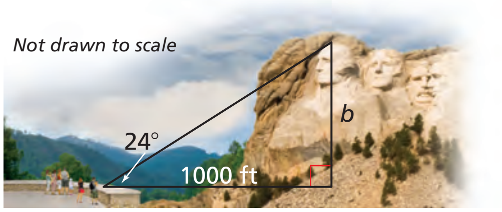


42. **MODELING WITH MATHEMATICS** Katoomba Scenic Railway in Australia is the steepest railway in the world. The railway makes an angle of about 52° with the ground. The railway extends horizontally about 458 feet. What is the height of the railway?

43. **MODELING WITH MATHEMATICS** A person whose eye level is 1.5 meters above the ground is standing 75 meters from the base of the Jin Mao Building in Shanghai, China. The person estimates the angle of elevation to the top of the building is about 80°. What is the approximate height of the building? *(See Example 6.)*

44. **MODELING WITH MATHEMATICS** The Duquesne Incline in Pittsburgh, Pennsylvania, has an angle of elevation of 30°. The track has a length of about 800 feet. Find the height of the incline.

45. **MODELING WITH MATHEMATICS** You are standing on the Grand View Terrace viewing platform at Mount Rushmore, 1000 feet from the base of the monument.

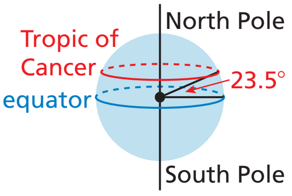


a. You look up at the top of Mount Rushmore at an angle of 24°. How high is the top of the monument from where you are standing? Assume your eye level is 5.5 feet above the platform.

b. The elevation of the Grand View Terrace is 5280 feet. Use your answer in part (a) to find the elevation of the top of Mount Rushmore.

46. **WRITING** Write a real-life problem that can be solved using a right triangle. Then solve your problem.

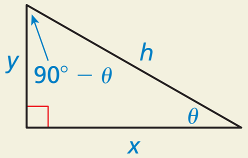
47. MATHEMATICAL CONNECTIONS The Tropic of Cancer is the circle of latitude farthest north of the equator where the Sun can appear directly overhead. It lies 23.5° north of the equator, as shown.



a. Find the circumference of the Tropic of Cancer using 3960 miles as the approximate radius of Earth.

b. What is the distance between two points on the Tropic of Cancer that lie directly across from each other?

48. HOW DO YOU SEE IT? Use the figure to answer each question.

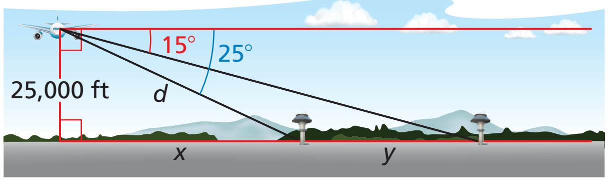


a. Which side is adjacent to θ?

b. Which side is opposite of θ?

c. Does $\cos \theta = \sin(90° - \theta)$? Explain.

49. PROBLEM SOLVING A passenger in an airplane sees two towns directly to the left of the plane.



a. What is the distance d from the airplane to the first town?

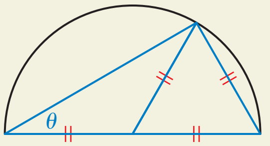
b. What is the horizontal distance x from the airplane to the first town?

c. What is the distance y between the two towns? Explain the process you used to find your answer.

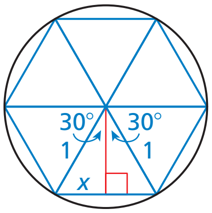
50. PROBLEM SOLVING You measure the angle of elevation from the ground to the top of a building as 32°. When you move 50 meters closer to the building, the angle of elevation is 53°. What is the height of the building?

51. MAKING AN ARGUMENT Your friend claims it is possible to draw a right triangle so the values of the cosine function of the acute angles are equal. Is your friend correct? Explain your reasoning.

52. THOUGHT PROVOKING Consider a semicircle with a radius of 1 unit, as shown below. Write the values of the six trigonometric functions of the angle θ. Explain your reasoning.



53. CRITICAL THINKING A procedure for approximating π based on the work of Archimedes is to inscribe a regular hexagon in a circle.



a. Use the diagram to solve for x. What is the perimeter of the hexagon?

b. Show that a regular n-sided polygon inscribed in a circle of radius 1 has a perimeter of $2n \cdot \sin\left(\frac{180}{n}\right)^{\circ}$.

c. Use the result from part (b) to find an expression in terms of n that approximates π. Then evaluate the expression when $n = 50$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the indicated conversion. *(Skills Review Handbook)*

54. 5 years to seconds

55. 12 pints to gallons

56. 5.6 meters to millimeters

Find the circumference and area of the circle with the given radius or diameter. *(Skills Review Handbook)*

57. $r = 6$ centimeters

58. $r = 11$ inches

59. $d = 14$ feet