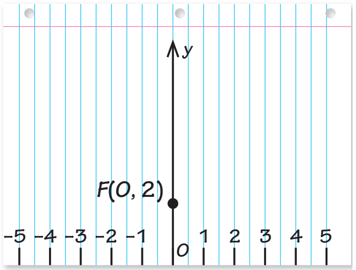
2.3 Focus of a Parabola

GO DIGITAL

Learning Target Graph and write equations of parabolas.

Success Criteria
- I can explain the relationships among the focus, the directrix, and the graph of a parabola.
- I can graph parabolas.
- I can write equations of parabolas.

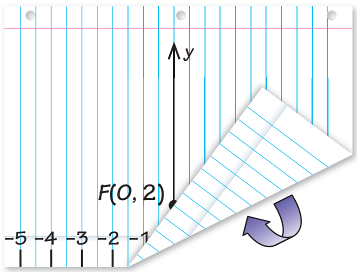
EXPLORE IT! Analyzing Graphs of Parabolas

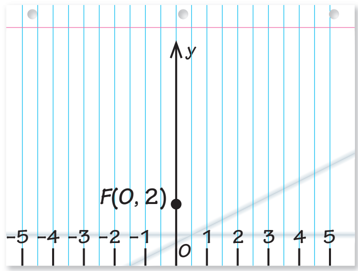


Work with a partner. Use dashes along the bottom of a piece of lined paper to mark and number equidistant points from −5 to 5 as shown. These dashes represent the units along the x-axis. Plot a point $F(0, 2)$ that is two units above 0. Draw a line through F to represent the y-axis.

a. Fold the paper so the point 0 (bottom of page) is on top of the plotted point. Unfold the paper and describe the line represented by the fold you made.

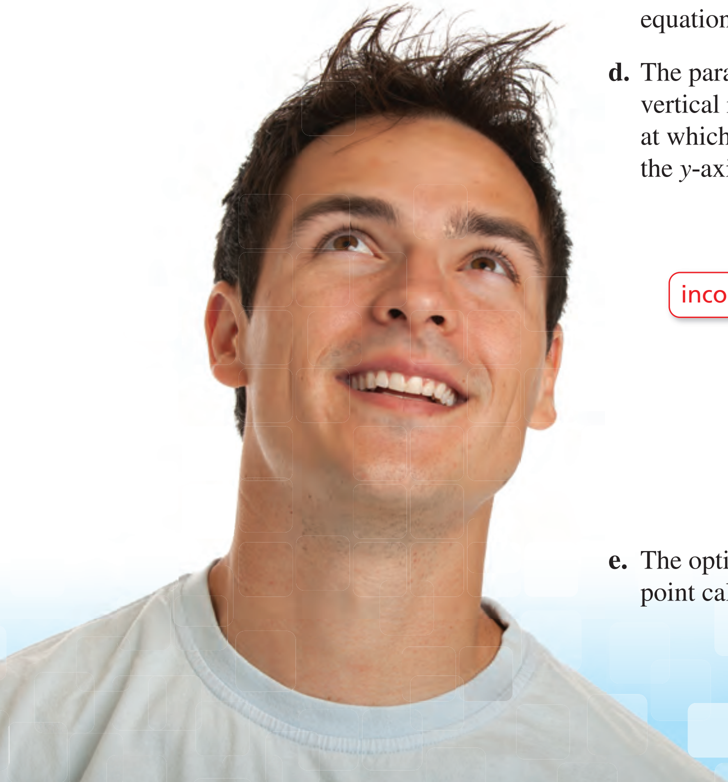
b. Repeat the process in part (a) with the points 1 and −1, 2 and −2, and so on. The diagrams below show the fold for the point 1. After you are done, examine the folds. What do you notice?





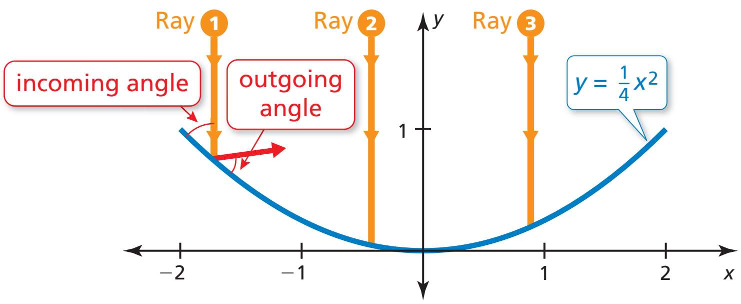
Math Practice

Construct Arguments
How does the shape of a parabola change as you move the focus closer and farther away from the vertex of the parabola?



c. On each fold for the points 0, 1, and 2, use the Pythagorean Theorem to find and label a point (x, y) that is equidistant from F and the x-axis. Then find an equation that represents the curve that passes through these points.

d. The parabola below represents the cross section of a satellite dish. When vertical rays enter the dish and hit the parabola, they reflect at the same angle at which they entered as shown. Draw the reflected rays so that they intersect the y-axis. What do you notice?



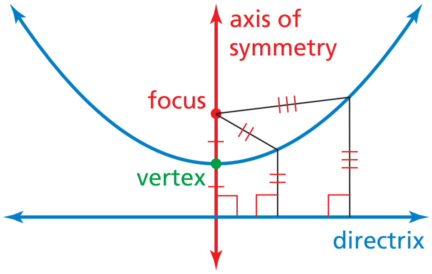
e. The optimal location for the receiver of the satellite dish in part (d) is at a point called the *focus* of the parabola. Determine the location of the focus.

Exploring the Focus and Directrix

Vocabulary
focus, *p. 64*
directrix, *p. 64*

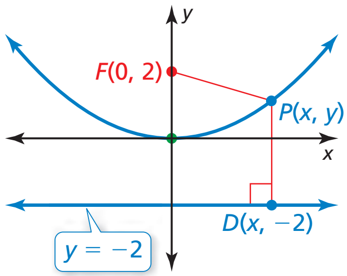
Previously, you learned that the graph of a quadratic function is a parabola that opens up or down. A parabola can also be defined as the set of all points (x, y) in a plane that are equidistant from a fixed point called the **focus** and a fixed line called the **directrix**.

- The focus is in the interior of the parabola and lies on the axis of symmetry.
- The vertex lies halfway between the focus and the directrix.
- The directrix is perpendicular to the axis of symmetry.



EXAMPLE 1 Deriving an Equation

Write an equation of the parabola with focus $F(0, 2)$ and directrix $y = -2$.



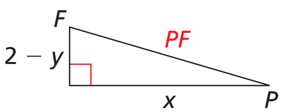
STUDY TIP
The distance from a point to a line is defined as the length of the perpendicular segment from the point to the line.

SOLUTION

The vertex is halfway between the focus and the directrix, at $(0, 0)$. Notice the line segments drawn from point F to point P and from point P to point D. By the definition of a parabola, these line segments, PD and PF, must be congruent.

$PD = PF$	Definition of a parabola
$\lvert y - (-2) \rvert = \sqrt{x^2 + (2 - y)^2}$	Write expressions for the lengths of the line segments.
$(y + 2)^2 = x^2 + (2 - y)^2$	Square each side.
$y^2 + 4y + 4 = x^2 + 4 - 4y + y^2$	Expand.
$8y = x^2$	Combine like terms.
$y = \frac{1}{8}x^2$	Divide each side by 8.

REMEMBER
You can use the Pythagorean Theorem to find distances in the coordinate plane.

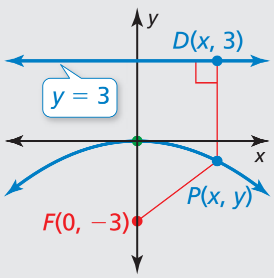


▶ So, an equation of the parabola is $y = \frac{1}{8}x^2$.

SELF-ASSESSMENT

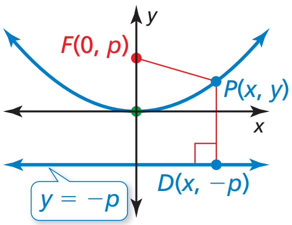
1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. Write an equation of the parabola with focus $F(0, -3)$ and directrix $y = 3$.



2. **WRITING** Explain how to find the coordinates of the focus of a parabola with vertex $(0, 0)$ and directrix $y = 5$.

You can derive the equation of a parabola that opens up or down with vertex (0, 0), focus (0, p), and directrix $y = -p$ using the procedure in Example 1.



$$|y - (-p)| = \sqrt{x^2 + (p - y)^2}$$
$$(y + p)^2 = x^2 + (p - y)^2$$
$$y^2 + 2py + p^2 = x^2 + p^2 - 2py + y^2$$
$$4py = x^2$$
$$y = \frac{1}{4p}x^2$$

The focus and directrix each lie $|p|$ units from the vertex. Parabolas can also open left or right, in which case the equation has the form $x = \frac{1}{4p}y^2$ when the vertex is (0, 0).

KEY IDEA

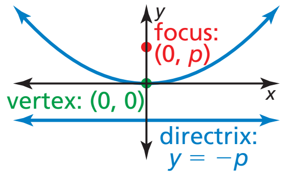
Standard Equations of a Parabola with Vertex at the Origin

Vertical axis of symmetry ($x = 0$)

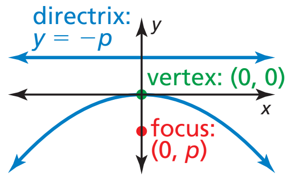
Equation: $y = \frac{1}{4p}x^2$

Focus: $(0, p)$

Directrix: $y = -p$



$p > 0$



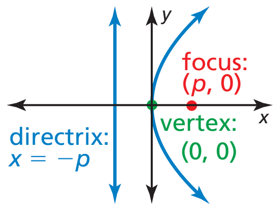
$p < 0$

Horizontal axis of symmetry ($y = 0$)

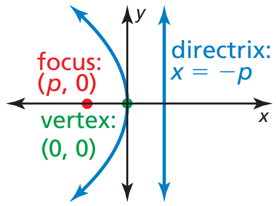
Equation: $x = \frac{1}{4p}y^2$

Focus: $(p, 0)$

Directrix: $x = -p$



$p > 0$



$p < 0$

Math Practice

Look for Structure

How does changing the value of p affect the graph of $y = \frac{1}{4p}x^2$?

EXAMPLE 2 Graphing an Equation of a Parabola

WATCH

Identify the focus, directrix, and axis of symmetry of $-4x = y^2$. Graph the equation.

SOLUTION

Step 1 Rewrite the equation in standard form.

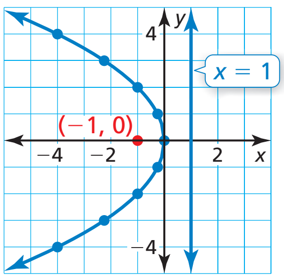
$-4x = y^2$ Write the original equation.

$x = -\frac{1}{4}y^2$ Divide each side by −4.

Step 2 Identify the focus, directrix, and axis of symmetry. The equation has the form $x = \frac{1}{4p}y^2$, where $p = -1$. The focus is $(p, 0)$, or $(-1, 0)$. The directrix is $x = -p$, or $x = 1$. Because y is squared, the axis of symmetry is the x-axis.

Step 3 Use a table of values to graph the equation. Notice that it is easier to substitute y-values and solve for x. Opposite y-values result in the same x-value.

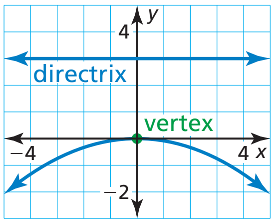
y	x
0	0
±1	−0.25
±2	−1
±3	−2.25
±4	−4



Writing Equations of Parabolas

EXAMPLE 3 Writing an Equation of a Parabola

WATCH



Write an equation of the parabola shown.

SOLUTION

Because the vertex is at the origin and the axis of symmetry is vertical, the equation has the form $y = \frac{1}{4p}x^2$. The directrix is $y = -p = 3$, so $p = -3$. Substitute -3 for p to write an equation of the parabola.

$$y = \frac{1}{4(-3)}x^2 = -\frac{1}{12}x^2$$

▶ So, an equation of the parabola is $y = -\frac{1}{12}x^2$.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Identify the focus, directrix, and axis of symmetry of the parabola. Graph the equation.

3. $y = 0.5x^2$ **4.** $-y = x^2$ **5.** $y^2 = 10x$

Write an equation of the parabola with vertex (0, 0) and the given directrix or focus.

6. directrix: $x = -3$ **7.** focus: $(-2, 0)$ **8.** focus: $\left(0, \frac{3}{2}\right)$

9. MP **REASONING** Which parabolas in Examples 2 and 3 are functions? Explain.

The vertex of a parabola is not always at the origin. As in previous transformations, adding a value to the input or output translates the graph.

KEY IDEA

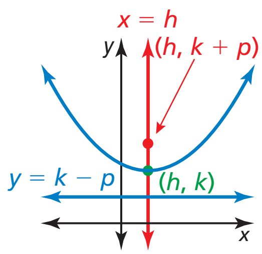
Standard Equations of a Parabola with Vertex at (*h*, *k*)

Vertical axis of symmetry ($x = h$)

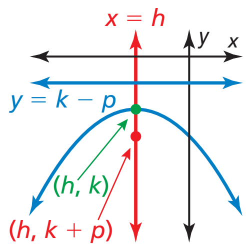
Equation: $y = \frac{1}{4p}(x - h)^2 + k$

Focus: $(h, k + p)$

Directrix: $y = k - p$



$p > 0$



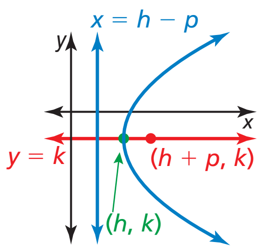
$p < 0$

Horizontal axis of symmetry ($y = k$)

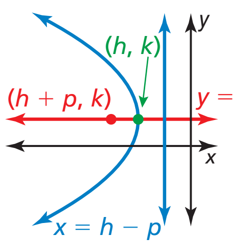
Equation: $x = \frac{1}{4p}(y - k)^2 + h$

Focus: $(h + p, k)$

Directrix: $x = h - p$



$p > 0$

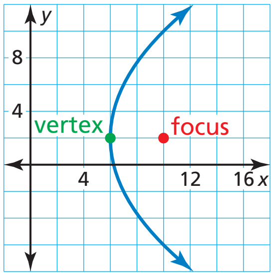


$p < 0$

EXAMPLE 4 Writing an Equation of a Translated Parabola

WATCH

GO DIGITAL



Write an equation of the parabola shown.

SOLUTION

Because the vertex is not at the origin and the axis of symmetry is horizontal, the equation has the form $x = \frac{1}{4p}(y - k)^2 + h$. The vertex (h, k) is $(6, 2)$ and the focus $(h + p, k)$ is $(10, 2)$, so $h = 6$, $k = 2$, and $p = 4$. Substitute these values to write an equation of the parabola.

$$x = \frac{1}{4(4)}(y - 2)^2 + 6 = \frac{1}{16}(y - 2)^2 + 6$$

▶ So, an equation of the parabola is $x = \frac{1}{16}(y - 2)^2 + 6$.

Solving Real-Life Problems

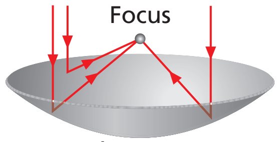


Diagram 1

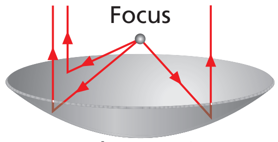


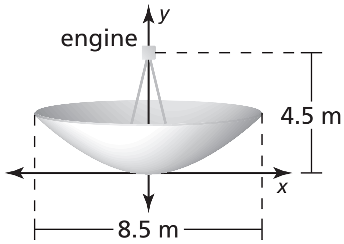
Diagram 2

Parabolic reflectors have cross sections that are parabolas to reflect sound, light, or other energy. Waves that hit a parabolic reflector parallel to the axis of symmetry are directed to the focus (Diagram 1). Similarly, waves that come from the focus and then hit the parabolic reflector are directed parallel to the axis of symmetry (Diagram 2).

EXAMPLE 5 Modeling Real Life

WATCH STEM

An electricity-generating dish uses a parabolic reflector to concentrate sunlight onto a high-frequency engine located at the focus of the reflector. The sunlight heats helium to 650°C to power the engine. Write an equation that represents the cross section of the dish shown with its vertex at (0, 0). What is the depth of the dish?



SOLUTION

Because the vertex is at the origin, and the axis of symmetry is vertical, the equation has the form $y = \frac{1}{4p}x^2$. The engine is at the focus, which is 4.5 meters above the vertex. So, $p = 4.5$. Substitute 4.5 for p to write the equation.

$$y = \frac{1}{4(4.5)}x^2 = \frac{1}{18}x^2$$

The depth of the dish is the y-value at the dish's outside edge. The dish extends $\frac{8.5}{2} = 4.25$ meters to either side of the vertex (0, 0), so find y when $x = 4.25$.

$$y = \frac{1}{18}(4.25)^2 \approx 1$$

▶ The depth of the dish is about 1 meter.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

10. Write an equation of a parabola with vertex $(-1, 4)$ and focus $(-1, 2)$.

11. A parabolic microwave antenna is 16 feet in diameter. Write an equation that represents the cross section of the antenna with its vertex at (0, 0) and its focus 10 feet to the right of the vertex. What is the depth of the antenna?

2.3 Practice WITH CalcChat® AND CalcView®

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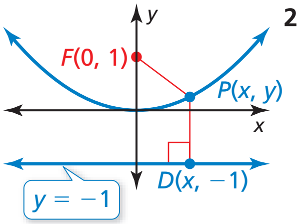
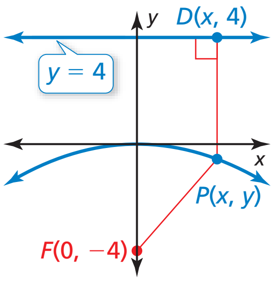
In Exercises 1–8, write an equation of the parabola.
▶ *Example 1*

1.



2.



3. focus: $(0, -2)$
directrix: $y = 2$

4. directrix: $y = 7$
focus: $(0, -7)$

5. vertex: $(0, 0)$
directrix: $y = -6$

6. vertex: $(0, 0)$
focus: $(0, 5)$

7. vertex: $(0, 0)$
focus: $(0, -10)$

8. vertex: $(0, 0)$
directrix: $y = -9$

In Exercises 9–16, identify the focus, directrix, and axis of symmetry of the parabola. Graph the equation.
▶ *Example 2*

9. $y = \frac{1}{8}x^2$

10. $y = -\frac{1}{12}x^2$

11. $x = -\frac{1}{20}y^2$

12. $x = \frac{1}{24}y^2$

13. $y^2 = 16x$

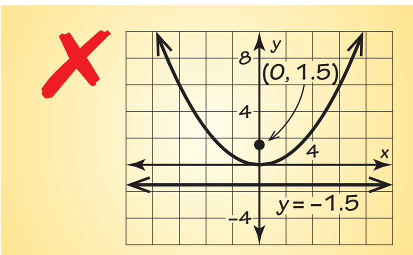
14. $-x^2 = 48y$

15. $6x^2 + 3y = 0$

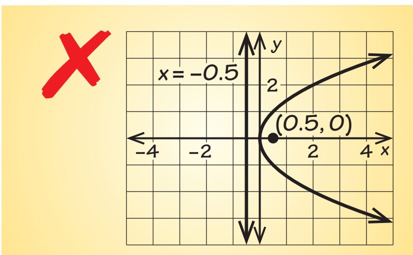
16. $8x^2 - y = 0$

ERROR ANALYSIS **In Exercises 17 and 18, describe and correct the error in graphing the equation.**

17. $-6x + y^2 = 0$



18. $0.5y^2 + x = 0$

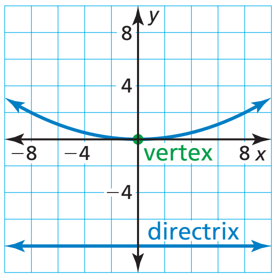


19. ANALYZING EQUATIONS The cross section (with units in inches) of a parabolic satellite dish can be modeled by the equation $y = \frac{1}{38}x^2$. How far is the focus from the vertex of the cross section? Explain.

20. ANALYZING EQUATIONS The cross section (with units in inches) of a parabolic spotlight can be modeled by the equation $x = \frac{1}{20}y^2$. How far is the focus from the vertex of the cross section? Explain.

In Exercises 21–24, write an equation of the parabola.
▶ *Example 3*

21.



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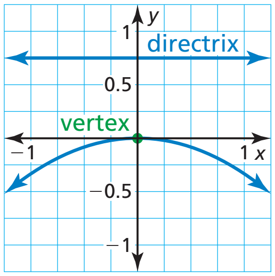

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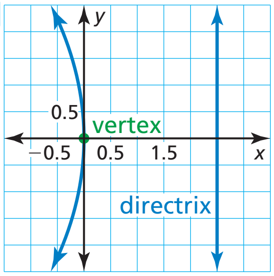

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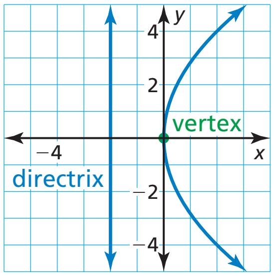


In Exercises 25–32, write an equation of the parabola with the given characteristics.

25. focus: $(-7, 0)$
directrix: $x = 7$

26. focus: $\left(\frac{2}{3}, 0\right)$
directrix: $x = -\frac{2}{3}$

27. directrix: $x = -10$
vertex: $(0, 0)$

28. directrix: $y = \frac{8}{3}$
vertex: $(0, 0)$

29. focus: $\left(0, -\frac{5}{3}\right)$
directrix: $y = \frac{5}{3}$

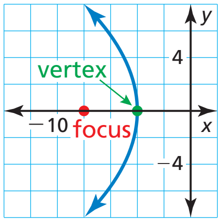
30. focus: $\left(0, \frac{5}{4}\right)$
directrix: $y = -\frac{5}{4}$

31. focus: $\left(0, \frac{6}{7}\right)$
vertex: $(0, 0)$

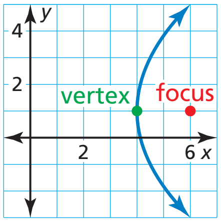
32. focus: $\left(-\frac{4}{5}, 0\right)$
vertex: $(0, 0)$

In Exercises 33–36, write an equation of the parabola. ▶ *Example 4*

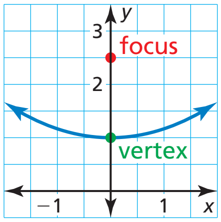
33.



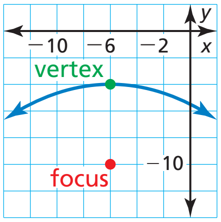
34.



35.



36.



In Exercises 37–40, write an equation of the parabola with the given characteristics.

37. directrix: $y = 12$
vertex: $(2, 3)$

38. directrix: $x = 4$
vertex: $(-7, -5)$

39. focus: $\left(\frac{5}{4}, -1\right)$
directrix: $x = \frac{3}{4}$

40. focus: $\left(-3, \frac{11}{2}\right)$
directrix: $y = -\frac{3}{2}$

In Exercises 41–46, identify the vertex, focus, directrix, and axis of symmetry of the parabola. Describe the transformations of the graph of the standard equation with $p = 1$ and vertex $(0, 0)$.

41. $y = \frac{1}{8}(x - 3)^2 + 2$

42. $y = -\frac{1}{4}(x + 2)^2 + 1$

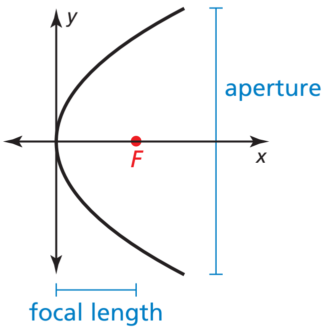
43. $x = \frac{1}{16}(y - 3)^2 + 1$

44. $y = (x + 3)^2 - 5$

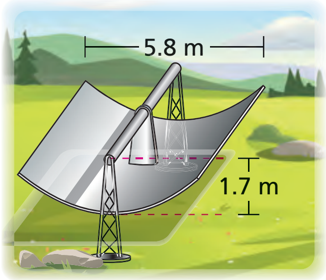
45. $x = -3(y + 4)^2 + 2$

46. $x = 4(y + 5)^2 - 1$

47. **MODELING REAL LIFE** Scientists studying dolphin echolocation simulate a bottlenose dolphin's clicking sounds using computer models. The sounds originate at the focus of a parabolic reflector. The parabola represents the cross section of the reflector with focal length of 1.3 inches and aperture width of 8 inches. Write an equation of the parabola. What is the depth of the reflector? ▶ *Example 5*



48. **MODELING REAL LIFE** Solar energy can be concentrated using a long trough that has a parabolic cross section, as shown in the figure. Write an equation that represents the cross section of the trough with its vertex at $(0, 0)$. What are the domain and range in this situation? What do they represent?



GO DIGITAL

49. **COLLEGE PREP** Which of the given characteristics describe parabolas that open down? Select all that apply.

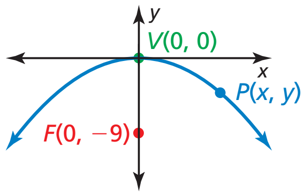
Ⓐ focus: $(0, -6)$
directrix: $y = 6$

Ⓑ focus: $(0, -2)$
directrix: $y = 2$

Ⓒ focus: $(0, 6)$
directrix: $y = -6$

Ⓓ focus: $(0, -1)$
directrix: $y = 1$

50. **COLLEGE PREP** Which of the following are possible coordinates of the point P in the graph shown? Select all that apply.

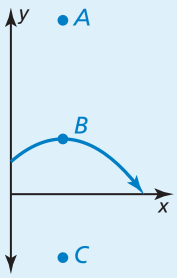


Ⓐ $(-6, -1)$ Ⓑ $\left(3, -\frac{1}{4}\right)$ Ⓒ $\left(4, -\frac{4}{9}\right)$

Ⓓ $\left(1, \frac{1}{36}\right)$ Ⓔ $(6, -1)$ Ⓕ $\left(2, -\frac{1}{18}\right)$

51. **ABSTRACT REASONING** As $|p|$ increases, how does the width of the graph of the equation $y = \frac{1}{4p}x^2$ change? Explain your reasoning.

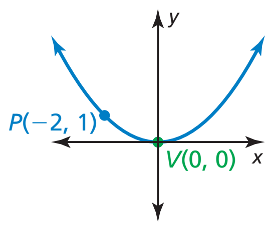
52. **HOW DO YOU SEE IT?** The graph shows the path of a volleyball served from an initial height of 6 feet as it travels over a net.



a. Label the vertex, focus, and a point on the directrix.

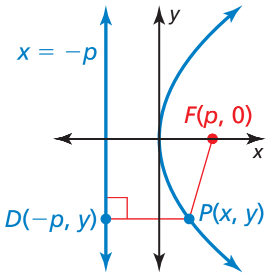
b. An underhand serve follows the same parabolic path but is hit from a height of 3 feet. How does this affect the focus? the directrix?

53. **CRITICAL THINKING** The distance from point P to the directrix is 2 units. Write an equation of the parabola.

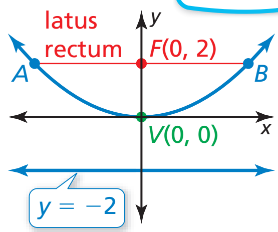


54. **THOUGHT PROVOKING** Two parabolas have the same focus (a, b). The distance from the vertex to the focus of each parabola is 2 units. Write an equation of each parabola. Identify the directrix of each parabola.

55. **MP REPEATED REASONING** Derive the equation of a parabola that opens to the right with vertex $(0, 0)$, focus $(p, 0)$, and directrix $x = -p$.

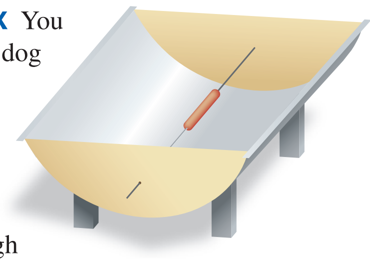


56. **DIG DEEPER** The *latus rectum* of a parabola is the line segment that is parallel to the directrix, passes through the focus, and has endpoints that lie on the parabola. Find the length of the latus rectum of the parabola shown.



GO DIGITAL

57. **PERFORMANCE TASK** You can make a solar hot dog cooker by shaping foil-lined poster board into a trough that has a parabolic cross section and passing a wire through each end piece. Design and construct your own hot dog cooker. Explain your process.



REVIEW & REFRESH

WATCH

In Exercises 58 and 59, write an equation of the line that passes through the given points.

58. $(1, -4), (2, -1)$

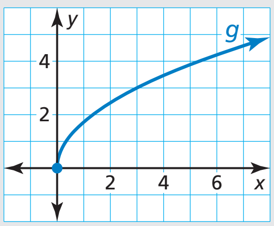
59. $(-3, 12), (0, 6)$

In Exercises 60 and 61, graph the inequality in a coordinate plane.

60. $y \leq -2.5$

61. $y > \frac{1}{2}x + 2$

62. The graph of square root function g is shown. Compare the average rate of change of g to the average rate of change of $h(x) = \sqrt[3]{\frac{3}{2}x}$ over the interval $x = 0$ to $x = 3$.



63. Write an equation of a parabola with vertex $(-2, -6)$ and focus $(-2, -1)$.

64. Use technology to find an equation of the line of best fit for the data.

x	0	3	6	7	11
y	4	9	24	29	46

65. **MP REASONING** A quadratic function has a minimum value at $x = -3$. When is the function increasing? decreasing?

66. Determine whether the table represents a *linear* or *nonlinear* function. Explain.

x	−1	1	3	5	7
y	6	3	−2	−9	−18

67. Let the graph of g be a translation 2 units up, followed by a reflection in the x-axis and a vertical stretch by a factor of 6 of the graph of $f(x) = x^2$. Write a rule for g and identify the vertex.

In Exercises 68–71, solve the equation.

68. $\frac{3}{5}n = -15$

69. $-2h + 17 = 12$

70. $3(2x - 4) = 3x + 9$

71. $4(2 - w) = -\frac{1}{2}(8w + 4)$

72. **MODELING REAL LIFE** You make a total of 11 one-point free throws, two-point shots, and three-point shots in a basketball game and score a total of 22 points. You make two more two-point shots than three-point shots. How many of each type of shot do you make?

73. Identify the focus, directrix, and axis of symmetry of $-3x = y^2$. Graph the equation.