11.1 Circumference and Arc Length

GO DIGITAL

Learning Target Understand circumference, arc length, and radian measure.

Success Criteria

- I can use the formula for the circumference of a circle to find measures.
- I can find arc lengths and use arc lengths to find measures.
- I can solve real-life problems involving circumference.
- I can explain radian measure and convert between degree and radian measure.

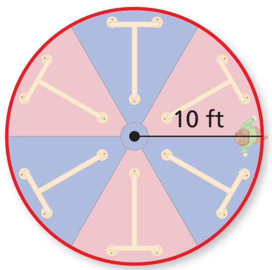
EXPLORE IT! Finding the Length of a Circular Arc

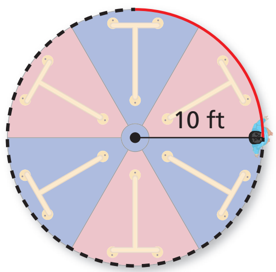
Work with a partner. A roundabout at a playground has a radius of 10 feet. As it rotates, a person can ride the roundabout at different distances from the center of the circular ride.

a. Find the distance that each person travels along the red circular arc.

i. one full rotation

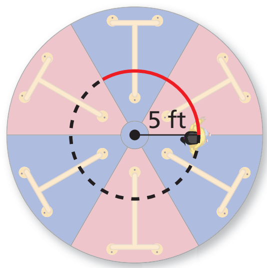
ii. one-fourth of a rotation

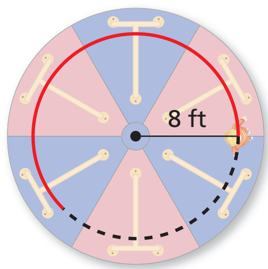




iii. one-third of a rotation

iv. five-eighths of a rotation





b. A person standing 8 feet from the center of the roundabout travels one-fourth of a rotation. Without performing any calculations, who travels farther, this person or the person in part (ii)? How do you know?

c. For what fraction of a rotation would a person standing 10 feet from the center of the roundabout travel the same distance as the person in part (iv)? Explain your reasoning.

d. Explain how to find the length of any circular arc.

Math Practice

Communicate Precisely
Explain how you can use arc measure to find the length of a circular arc. Write a formula to support your answer.

Using the Formula for Circumference

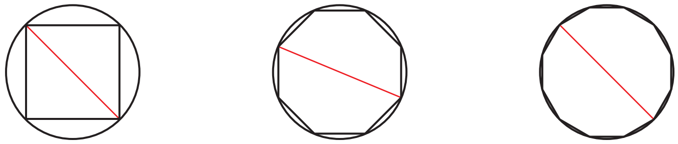
GO DIGITAL

Vocabulary

AZ VOCAB

circumference, *p. 582*
arc length, *p. 583*
radian, *p. 585*

The **circumference** of a circle is the distance around the circle. Consider a regular polygon inscribed in a circle. As the number of sides increases, the polygon approximates the circle, and the ratio of the perimeter of the polygon to the diameter of the circle approaches $\pi \approx 3.14159\ldots$.

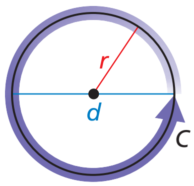


For all circles, the ratio of the circumference C to the diameter d is the same. This ratio is $\frac{C}{d} = \pi$. Solving for C yields the formula for the circumference of a circle, $C = \pi d$. Because $d = 2r$, where r is the radius, you can also write the formula as $C = \pi(2r) = 2\pi r$.

KEY IDEA

Circumference of a Circle

The circumference C of a circle is $C = \pi d$ or $C = 2\pi r$, where d is the diameter of the circle and r is the radius of the circle.



$C = \pi d = 2\pi r$

EXAMPLE 1 Using the Formula for Circumference

WATCH

Find each indicated measure.

a. circumference of a circle with a radius of 9 centimeters

b. radius of a circle with a circumference of 26 meters

STUDY TIP

You have used 3.14 to approximate the value of π. Throughout this chapter, use technology to evaluate expressions involving π.

SOLUTION

a.

$C = 2\pi r$	Write formula for circumference.
$= 2 \cdot \pi \cdot 9$	Substitute 9 for r.
$= 18\pi$	Multiply.
≈ 56.55	Use technology.

▶ The circumference is about 56.55 centimeters.

b.

$C = 2\pi r$	Write formula for circumference.
$26 = 2\pi r$	Substitute 26 for C.
$\frac{26}{2\pi} = r$	Divide each side by 2π.
$4.14 \approx r$	Use technology.

▶ The radius is about 4.14 meters.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. Find the circumference of a circle with a diameter of 5 inches.

2. Find the diameter of a circle with a circumference of 17 feet.

Finding and Using Arc Lengths

An **arc length** is a portion of the circumference of a circle. You can use the measure of the arc (in degrees) to find its length (in linear units).

GO DIGITAL

STUDY TIP

Just as the terms *point*, *line*, and *plane* are undefined, the distance around a circular arc is another example of an undefined geometric term.

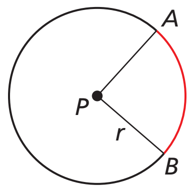
KEY IDEA

Arc Length

In a circle, the ratio of the length of a given arc to the circumference is equal to the ratio of the measure of the arc to 360°.

$$\frac{\text{Arc length of } \widehat{AB}}{2\pi r} = \frac{m\widehat{AB}}{360^\circ}, \text{ or}$$

$$\text{Arc length of } \widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$$

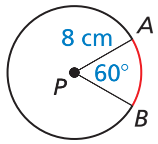


EXAMPLE 2 Finding and Using Arc Lengths

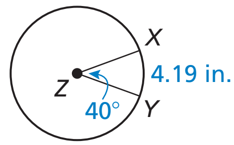
WATCH

Find each indicated measure.

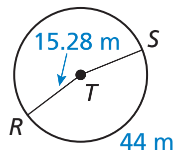
a. arc length of $\widehat{AB}$



b. circumference of $\odot Z$



c. $m\widehat{RS}$



SOLUTION

a. $\text{Arc length of } \widehat{AB} = \frac{60^\circ}{360^\circ} \cdot 2\pi(8)$

$\approx 8.38 \text{ cm}$

b. $\frac{\text{Arc length of } \widehat{XY}}{C} = \frac{m\widehat{XY}}{360^\circ}$

$\frac{4.19}{C} = \frac{40^\circ}{360^\circ}$

$\frac{4.19}{C} = \frac{1}{9}$

$37.71 \text{ in.} = C$

c. $\frac{\text{Arc length of } \widehat{RS}}{2\pi r} = \frac{m\widehat{RS}}{360^\circ}$

$\frac{44}{2\pi(15.28)} = \frac{m\widehat{RS}}{360^\circ}$

$360^\circ \cdot \frac{44}{2\pi(15.28)} = \frac{m\widehat{RS}}{360^\circ} \cdot 360^\circ$

$165^\circ \approx m\widehat{RS}$

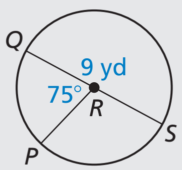
SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

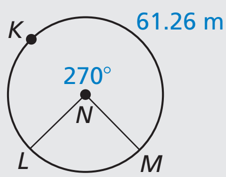
3. **WRITING** Describe the difference between an arc measure and an arc length.

Find the indicated measure.

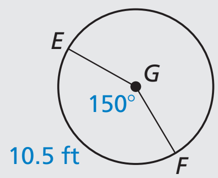
4. arc length of $\widehat{PQ}$



5. circumference of $\odot N$



6. radius of $\odot G$



GO DIGITAL

Solving Real-Life Problems

EXAMPLE 3 Using Circumference to Find Distance Traveled

WATCH INFO

The dimensions of a car tire are shown. How many feet does the tire travel when it makes 15 revolutions?

SOLUTION

Step 1 Find the diameter of the tire.

$$d = 15 + 2(5.5) = 26 \text{ in.}$$

Step 2 Find the circumference of the tire.

$$C = \pi d = \pi \cdot 26 = 26\pi \text{ in.}$$

Step 3 Find the distance the tire travels in 15 revolutions. In one revolution, the tire travels a distance equal to its circumference. So, in 15 revolutions, the tire travels a distance equal to 15 times its circumference.

Distance traveled = Number of revolutions • Circumference

$$= 15 \cdot 26\pi$$

$$\approx 1225.2 \text{ in.}$$

COMMON ERROR

Always pay attention to units. In Example 3, you need to convert units to obtain the correct answer.

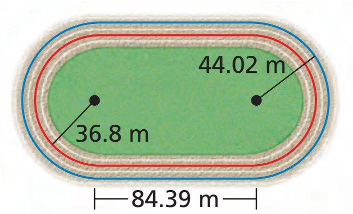
Step 4 Use unit analysis. Change 1225.2 inches to feet.

$$1225.2 \text{ in.} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} = 102.1 \text{ ft}$$

▶ The tire travels about 102 feet.

EXAMPLE 4 Using Arc Length to Find Distances

WATCH



The curves at the ends of the track shown are 180° arcs of circles. The radius of the arc for a runner on the inner path shown in the diagram is 36.8 meters. About how far does the runner travel in one lap?

SOLUTION

The path of the runner is made of two straight sections and two semicircles. To find the total distance, find the sum of the lengths of each part.

Distance = 2 • Length of each straight section + 2 • Length of each semicircle

$$= 2(84.39) + 2\left(\tfrac{1}{2} \cdot 2\pi \cdot 36.8\right)$$

$$\approx 400.0$$

▶ The runner travels about 400 meters.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

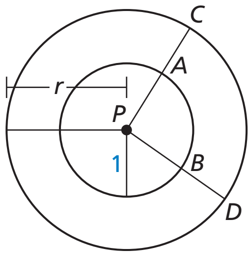
7. A car tire has a diameter of 28 inches. How many revolutions does the tire make while traveling 500 feet?

8. In Example 4, the radius of the arc for a runner on the outer path is 44.02 meters. The runner completes one lap. Without performing any calculations, how do you know which runner travels farther? Calculate how much farther this runner travels. Explain your reasoning.

Measuring Angles in Radians

GO DIGITAL

In a circle of radius 1, the *radian* measure of a given central angle can be thought of as the length of the arc associated with the angle. The radian measure of a complete circle (360°) is exactly 2π radians, because the circumference of a circle of radius 1 is exactly 2π. You can use this fact to convert from degree measure to radian measure and vice versa.



In a circle, you now know the ratio of the length of a given arc to the circumference is equal to the ratio of the measure of the arc to 360°. To see why, consider the diagram.

Because a circle of radius 1 has circumference 2π, the arc length of $\widehat{AB}$ is $\dfrac{m\widehat{AB}}{360^\circ} \cdot 2\pi$.

Recall that two arcs are similar arcs if and only if they have the same measure. Because $m\widehat{CD} = m\widehat{AB}$, $\widehat{CD}$ and $\widehat{AB}$ are similar. So, you can write the following proportion.

$$\frac{\text{Arc length of } \widehat{CD}}{\text{Arc length of } \widehat{AB}} = \frac{r}{1}$$

$$\text{Arc length of } \widehat{CD} = r \cdot \text{Arc length of } \widehat{AB}$$

$$\text{Arc length of } \widehat{CD} = r \cdot \frac{m\widehat{AB}}{360^\circ} \cdot 2\pi$$

This form of the equation shows that the arc length associated with central angle *CPD* is *proportional to the radius* of the circle. The constant of proportionality, $\dfrac{m\widehat{AB}}{360^\circ} \cdot 2\pi$, is defined to be the **radian** measure of the central angle associated with the arc.

STUDY TIP

Radian measure is simply another measurement for angles. In some mathematical applications and formulas, calculations are made easier by using radians instead of degrees.

KEY IDEA

Converting between Degrees and Radians

Degrees to radians Multiply degree measure by $\dfrac{2\pi \text{ radians}}{360^\circ}$, or $\dfrac{\pi \text{ radians}}{180^\circ}$.

Radians to degrees Multiply radian measure by $\dfrac{360^\circ}{2\pi \text{ radians}}$, or $\dfrac{180^\circ}{\pi \text{ radians}}$.

EXAMPLE 5 Converting between Degree and Radian Measure

WATCH

a. Convert 45° to radians.

b. Convert $\dfrac{3\pi}{2}$ radians to degrees.

SOLUTION

a. $45^\circ \cdot \dfrac{\pi \text{ radians}}{180^\circ} = \dfrac{\pi}{4}$ radian

So, $45^\circ = \dfrac{\pi}{4}$ radian.

b. $\dfrac{3\pi}{2}$ radians $\cdot \dfrac{180^\circ}{\pi \text{ radians}} = 270^\circ$

So, $\dfrac{3\pi}{2}$ radians $= 270^\circ$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

9. Convert 15° to radians.

10. Convert $\dfrac{4\pi}{3}$ radians to degrees.

11.1 Practice WITH CalcChat® AND CalcView®

GO DIGITAL

In Exercises 1–8, find the indicated measure.
Examples 1 and 2

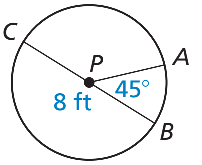
1. circumference of a circle with a radius of 6 inches

2. circumference of a circle with a diameter of 5 inches

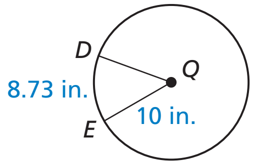
3. diameter of a circle with a circumference of 63 feet

4. radius of a circle with a circumference of 28π

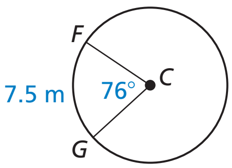
5. arc length of $\widehat{AB}$



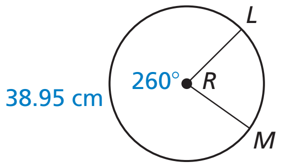
6. $m\widehat{DE}$



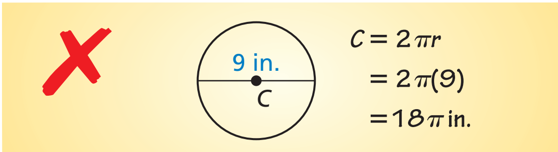
7. circumference of $\odot C$



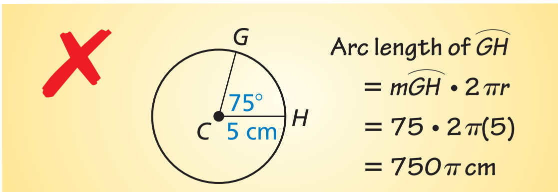
8. radius of $\odot R$



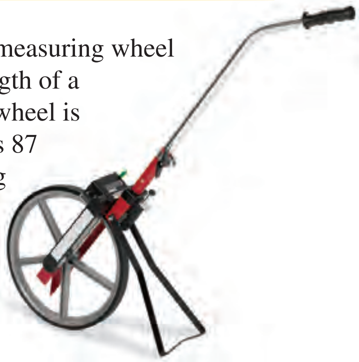
9. ERROR ANALYSIS Describe and correct the error in finding the circumference of $\odot C$.



10. ERROR ANALYSIS Describe and correct the error in finding the length of $\widehat{GH}$.



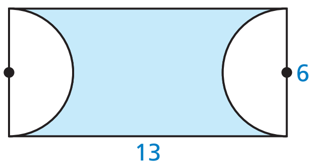
11. MODELING REAL LIFE A measuring wheel is used to calculate the length of a path. The diameter of the wheel is 8 inches. The wheel makes 87 complete revolutions along the length of the path. How many feet long is the path?
Example 3



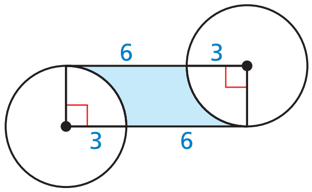
12. MODELING REAL LIFE You ride your bicycle 40 meters. How many complete revolutions does the front wheel make?

In Exercises 13–16, find the perimeter of the shaded region. *Example 4*

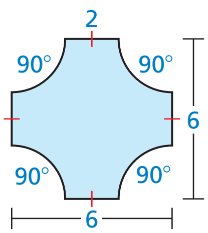
13.



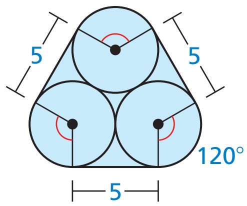
14.



15.



16.



In Exercises 17–20, convert the angle measure.
Example 5

17. Convert 70° to radians.

18. Convert 300° to radians.

19. Convert $\frac{11\pi}{12}$ radians to degrees.

20. Convert $\frac{\pi}{8}$ radian to degrees.

In Exercises 21 and 22, find the circumference of the circle represented by the given equation. Write the circumference in terms of π.

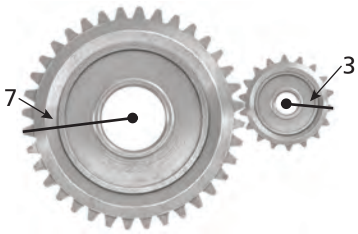
21. $x^2 + y^2 = 16$

22. $(x + 2)^2 + (y - 3)^2 = 9$

23. **MP STRUCTURE** A semicircle has endpoints $(-2, 5)$ and $(2, 8)$. Find the arc length of the semicircle.

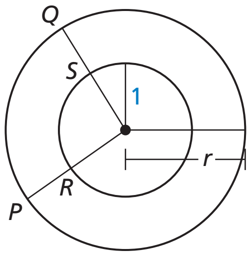
24. **MP REASONING** $\overset{\frown}{EF}$ is an arc on a circle with radius r. Let $x°$ be the measure of $\overset{\frown}{EF}$. Describe the effect on the length of $\overset{\frown}{EF}$ if you (a) double the radius of the circle, and (b) double the measure of $\overset{\frown}{EF}$.

25. **MODELING REAL LIFE** How many revolutions does the smaller gear complete during a single revolution of the larger gear?



26. **MODELING REAL LIFE** The London Eye is a Ferris wheel in London, England, with cars that travel 0.26 meter per second. How many minutes does it take the London Eye to complete one full revolution?

27. **MP STRUCTURE** Use the diagram to show that the length of $\overset{\frown}{PQ}$ is proportional to the radius r.

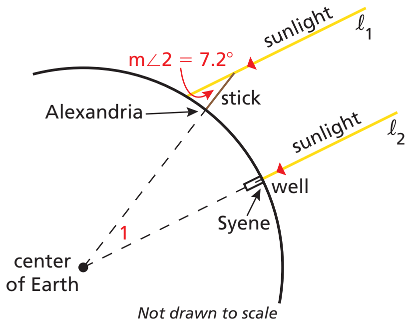


28. **COLLEGE PREP** A 45° arc in $\odot C$ and a 30° arc in $\odot P$ have the same length. What is the ratio of the radius r_1 of $\odot C$ to the radius r_2 of $\odot P$?

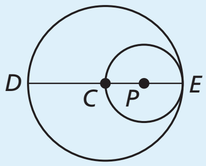
Ⓐ 1 to 12 Ⓑ 1 to 8

Ⓒ 2 to 3 Ⓓ 3 to 2

29. **MP PROBLEM SOLVING** Over 2000 years ago, the Greek scholar Eratosthenes estimated Earth's circumference by assuming that the Sun's rays were parallel. He chose a day when the Sun shone straight down into a well in the city of Syene. At noon, he measured the angle the Sun's rays made with a vertical stick in the city of Alexandria. Eratosthenes assumed that the distance from Syene to Alexandria was about 575 miles. Explain how Eratosthenes was able to use this information to estimate Earth's circumference. Then estimate Earth's circumference.



30. **HOW DO YOU SEE IT?**
Compare the circumference of $\odot P$ to the length of $\overset{\frown}{DE}$. Explain your reasoning.

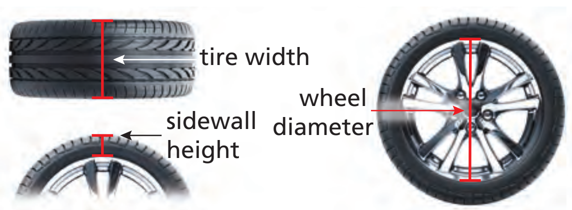


31. **MP STRUCTURE** Find the circumference of each circle.

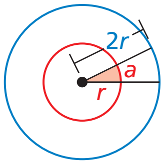
a. a circle circumscribed about a square with a side length of 6 centimeters

b. a circle inscribed in an equilateral triangle with a side length of 9 inches

32. **PERFORMANCE TASK** Tire sizes usually involve three numbers. Research and explain the meanings of these numbers. Then choose a tire and state its size and its warranty. Estimate the number of rotations for which the tire is covered by its warranty.



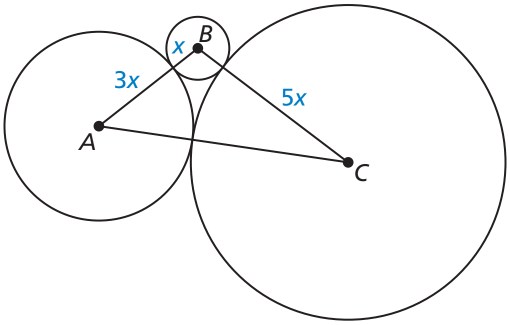
33. MAKING AN ARGUMENT In the diagram, the measure of the red shaded angle is 30°. The arc length a is 2. Your classmate claims that it is possible to find the circumference of the blue circle without finding the radius of either circle. Is your classmate correct? Explain your reasoning.



34. MODELING WITH MATHEMATICS What is the measure (in radians) of the angle formed by the hands of a clock at each time? Explain your reasoning.

a. 1:30 P.M. **b.** 3:15 P.M.

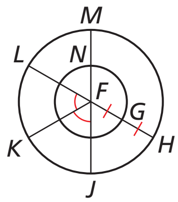
35. DIG DEEPER The sum of the circumferences of circles A, B, and C is 63π. Find AC.



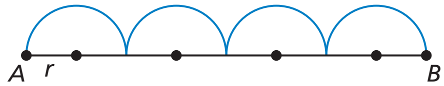
36. THOUGHT PROVOKING A circle is represented by the equation $(x + 7)^2 + (y - 4)^2 = 169$. The points $A(-19, 9)$ and $B(-2, 16)$ lie on the circle. Find the length of $\overset{\frown}{AB}$. Explain your reasoning.

GO DIGITAL

37. PROOF The circles in the diagram are concentric and $\overline{FG} \cong \overline{GH}$. Prove that $\overset{\frown}{JK}$ and $\overset{\frown}{NG}$ have the same length.



38. MP REPEATED REASONING $\overline{AB}$ is divided into congruent segments, each of which is the diameter of a semicircle.



a. What is the sum of the arc lengths?

b. What is the sum of the arc lengths when $\overline{AB}$ is divided into 8 congruent segments? 16 congruent segments? n congruent segments? Explain your reasoning.

REVIEW & REFRESH

WATCH

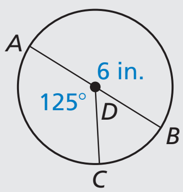
In Exercises 39 and 40, find the area of the polygon with the given vertices.

39. $X(2, 4)$, $Y(8, -1)$, $Z(2, -1)$

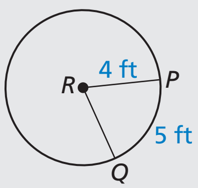
40. $L(-3, 1)$, $M(4, 1)$, $N(4, -5)$, $P(-3, -5)$

In Exercises 41 and 42, find the indicated measure.

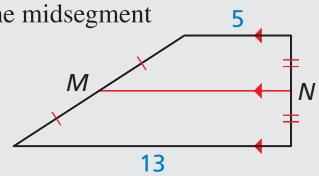
41. arc length of $\overset{\frown}{AC}$



42. $m\overset{\frown}{PQ}$



43. Find the length of the midsegment of the trapezoid.



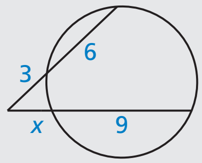
In Exercises 44 and 45, find the center and radius of the circle. Then graph the circle.

44. $x^2 + y^2 = 16$

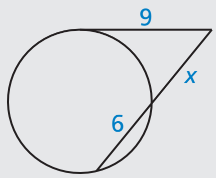
45. $(x - 3)^2 + (y - 7)^2 = 81$

In Exercises 46 and 47, find the value of x.

46.



47.



48. MODELING REAL LIFE L'Umbracle is an open-air gallery and garden in Valencia, Spain. The structure is composed of 55 parabolic arches. One of the arches can be represented by a parabola with focus (0, 5.5) and vertex (0, 18). Write an equation for the parabola.

49. Find the value of x that makes $\triangle PQR \sim \triangle XYZ$.