4.4 Congruence and Transformations

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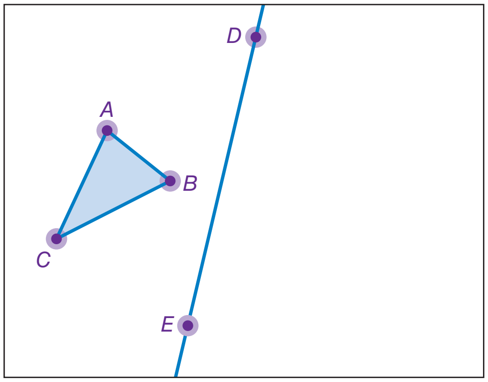
Learning Target Understand congruence transformations.

Success Criteria

- I can identify congruent figures.
- I can describe congruence transformations.
- I can use congruence transformations to solve problems.

EXPLORE IT! Reflecting Figures in Lines

Work with a partner. Use technology to draw any scalene triangle and label it $\triangle ABC$. Draw any line, $\overleftrightarrow{DE}$, and another line that is parallel to $\overleftrightarrow{DE}$.

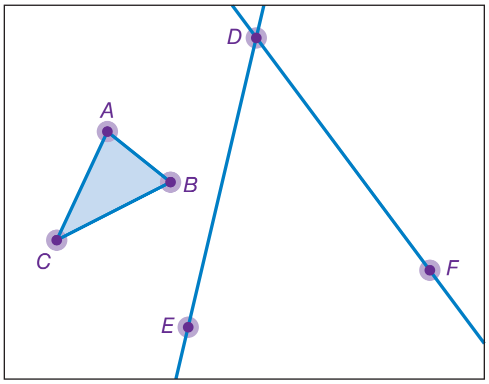


a. Reflect $\triangle ABC$ in $\overleftrightarrow{DE}$, followed by a reflection in the other line to form $\triangle A''B''C''$. What do you notice? Make several observations.

b. Is there a single transformation that maps $\triangle ABC$ to $\triangle A''B''C''$? Explain.

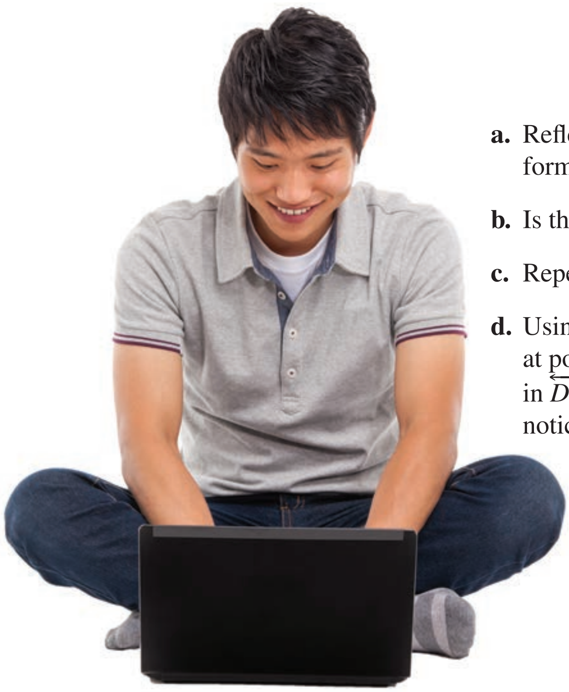
c. Repeat parts (a) and (b) with other figures. What do you notice?

d. Using the same triangle and line $\overleftrightarrow{DE}$, draw line $\overleftrightarrow{DF}$ that intersects $\overleftrightarrow{DE}$ at point D so that $\angle EDF$ is an acute or right angle. Then reflect $\triangle ABC$ in $\overleftrightarrow{DE}$, followed by a reflection in $\overleftrightarrow{DF}$ to form $\triangle A''B''C''$. What do you notice? Make several observations.



e. Is there a single transformation that maps $\triangle ABC$ to $\triangle A''B''C''$? Explain.

f. Repeat parts (d) and (e) with other figures. What do you notice?



Math Practice

Make Conjectures
What conjectures can you make based on your results? How can you test your conjectures?

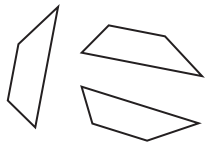
Identifying Congruent Figures

Vocabulary

congruent figures, *p. 192*
congruence transformation, *p. 193*

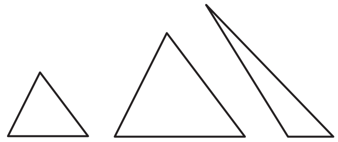
Two geometric figures are **congruent figures** if and only if there is a rigid motion or a composition of rigid motions that maps one of the figures to the other. Congruent figures have the same size and same shape.

Congruent



same size and same shape

Not congruent

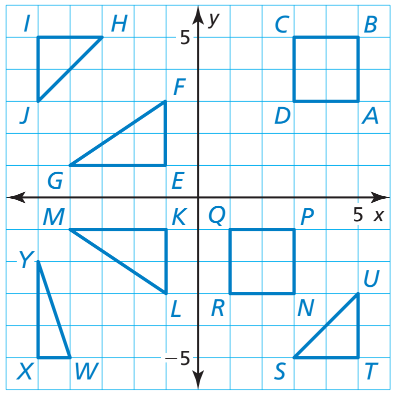


different sizes or shapes

You can identify congruent figures in the coordinate plane by identifying the rigid motion or composition of rigid motions that maps one of the figures to the other. Recall from Postulates 4.1–4.3 and Theorem 4.1 that translations, reflections, rotations, and compositions of these transformations are rigid motions.

EXAMPLE 1 Identifying Congruent Figures

Identify any congruent figures in the coordinate plane. Explain.



SOLUTION

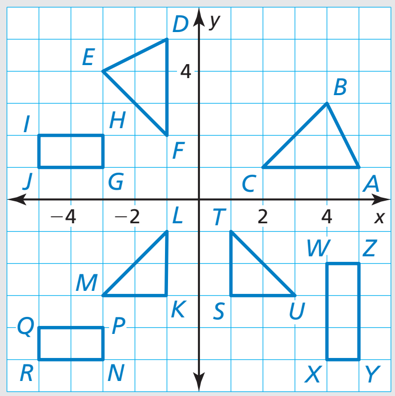
Square $NPQR$ is a translation of square $ABCD$ 2 units left and 6 units down. So, square $ABCD$ and square $NPQR$ are congruent.

$\triangle KLM$ is a reflection of $\triangle EFG$ in the x-axis. So, $\triangle EFG$ and $\triangle KLM$ are congruent.

$\triangle STU$ is a 180° rotation of $\triangle HIJ$. So, $\triangle HIJ$ and $\triangle STU$ are congruent.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

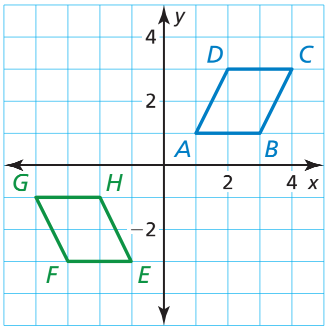
1. Identify any congruent figures in the coordinate plane. Explain.



Congruence Transformations

Another name for a rigid motion or a combination of rigid motions is a **congruence transformation** because the preimage and image are congruent. The terms *rigid motion* and *congruence transformation* are interchangeable.

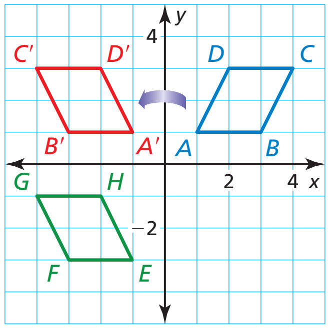
EXAMPLE 2 Describing a Congruence Transformation



Describe a congruence transformation that maps ▱*ABCD* to ▱*EFGH*.

SOLUTION

Two sides of ▱*ABCD* rise from left to right, and the corresponding sides of ▱*EFGH* fall from left to right. If you reflect ▱*ABCD* in the *y*-axis, as shown, then the image, ▱*A′B′C′D′*, will have the same orientation as ▱*EFGH*.



Then you can map ▱*A′B′C′D′* to ▱*EFGH* using a translation 4 units down.

▶ So, a congruence transformation that maps ▱*ABCD* to ▱*EFGH* is a reflection in the *y*-axis, followed by a translation 4 units down.

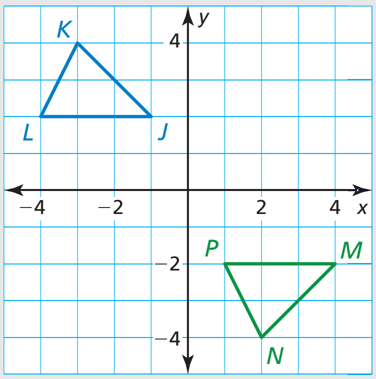
SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

2. **MP REASONING** A composition of rigid motions maps one figure to another figure. Is the image at each step of the composition congruent to the preimage and the final image? Explain.

3. In Example 2, describe another congruence transformation that maps ▱*ABCD* to ▱*EFGH*.

4. Describe a congruence transformation that maps △*JKL* to △*MNP*.



Using Theorems about Congruence Transformations

Compositions of two reflections result in either a translation or a rotation. A composition of two reflections in parallel lines results in a translation, as described in the following theorem.

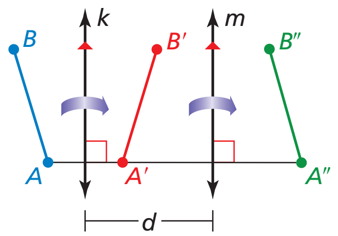
THEOREM

4.2 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If A'' is the image of A, then

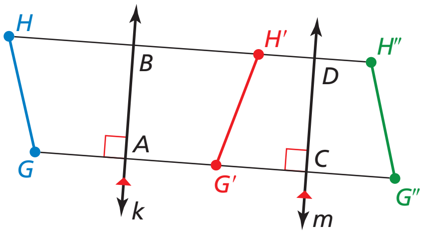
1. $\overline{AA''}$ is perpendicular to k and m, and
2. $AA'' = 2d$, where d is the distance between k and m.



Proof Exercise 32, page 198

EXAMPLE 3 Using the Reflections in Parallel Lines Theorem

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $H''D = 4$.



a. Name any segments congruent to each segment: $\overline{GH}$, $\overline{HB}$, and $\overline{GA}$.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?

SOLUTION

a. $\overline{GH} \cong \overline{G'H'}$, and $\overline{GH} \cong \overline{G''H''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.

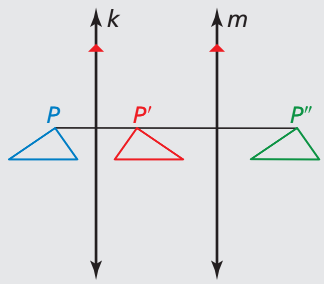
b. Yes, $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m. So, $ABDC$ is a rectangle because it has four right angles. $\overline{AC}$ and $\overline{BD}$ are opposite sides of rectangle $ABDC$, so $AC = BD$.

c. By the properties of reflections, $H'B = 9$ and $H'D = 4$. The Reflections in Parallel Lines Theorem implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4) = 26$ units.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Use the figure. The distance between line k and line m is 1.6 centimeters.



5. The preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure to the green figure.

6. What is the relationship between $\overline{PP'}$ and line k? Explain.

7. What is the distance between P and P''?

A composition of two reflections in intersecting lines results in a rotation, as described in the following theorem.

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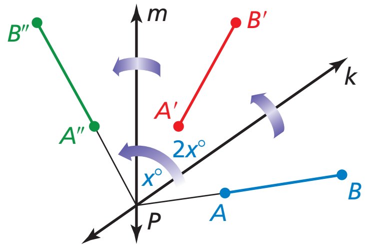
THEOREM

4.3 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P, then a reflection in line k followed by a reflection in line m is the same as a rotation about point P.

The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by lines k and m.

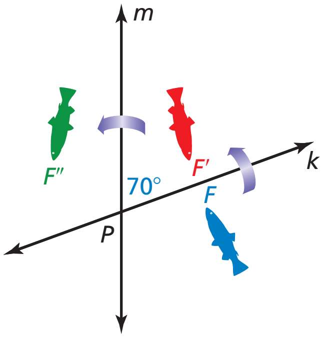


$m\angle BPB'' = 2x°$

Proof Exercise 22, page 242

EXAMPLE 4 Using the Reflections in Intersecting Lines Theorem

WATCH



In the diagram, the figure is reflected in line k. The image is then reflected in line m. Describe a single transformation that maps F to F''.

SOLUTION

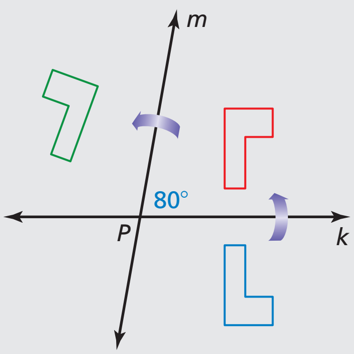
By the Reflections in Intersecting Lines Theorem, a reflection in line k followed by a reflection in line m is the same as a rotation about point P. The measure of the acute angle formed by lines k and m is 70°. So, by the Reflections in Intersecting Lines Theorem, the angle of rotation is $2(70°) = 140°$.

▶ A single transformation that maps F to F'' is a 140° rotation about point P. You can check that this is correct by tracing lines k and m and point F, then rotating the point 140°.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

8. In the diagram, the preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure to the green figure.



9. A rotation of 76° maps C to C'. To map C to C' using two reflections, what is the measure of the angle formed by the intersecting lines of reflection?

4.4 Practice WITH CalcChat® AND CalcView®

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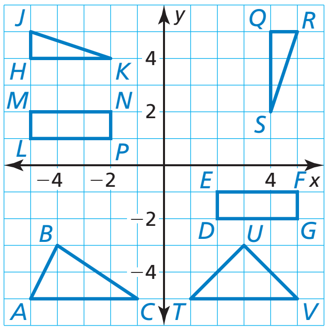
In Exercises 1 and 2, identify any congruent figures in the coordinate plane. Explain. *Example 1*

1.



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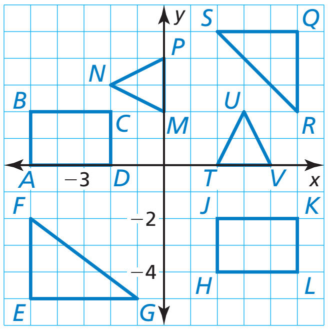
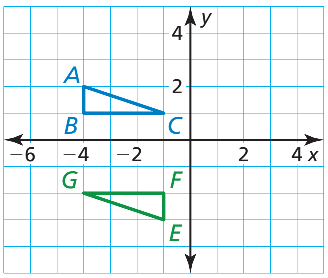


In Exercises 3 and 4, describe a congruence transformation that maps the blue preimage to the green image. *Example 2*

3.



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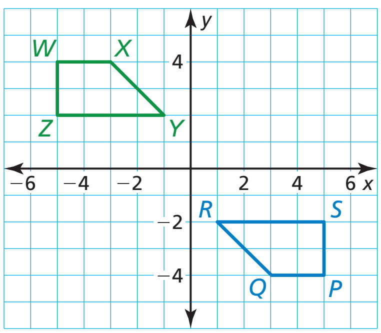


In Exercises 5–8, determine whether the polygons with the given vertices are congruent. Use transformations to explain your reasoning.

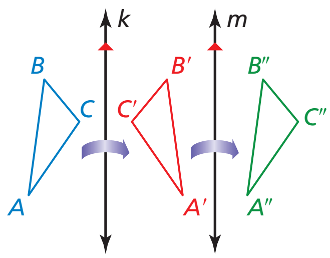
5. $Q(2, 4)$, $R(5, 4)$, $S(4, 1)$ and $T(6, 4)$, $U(9, 4)$, $V(8, 1)$

6. $W(-3, 1)$, $X(2, 1)$, $Y(4, -4)$, $Z(-5, -4)$ and $C(-1, -3)$, $D(-1, 2)$, $E(4, 4)$, $F(4, -5)$

7. $J(1, 1)$, $K(3, 2)$, $L(4, 1)$ and $M(6, 1)$, $N(5, 2)$, $P(2, 1)$

8. $A(0, 0)$, $B(1, 2)$, $C(4, 2)$, $D(3, 0)$ and $E(0, -5)$, $F(-1, -3)$, $G(-4, -3)$, $H(-3, -5)$

In Exercises 9–12, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m. *Example 3*



9. A translation maps $\triangle ABC$ to which triangle?

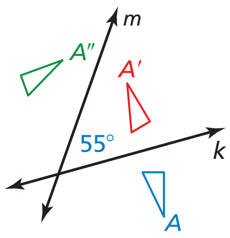
10. Which lines are perpendicular to $\overline{AA''}$?

11. If the distance between k and m is 2.6 inches, what is the length of $\overline{CC''}$?

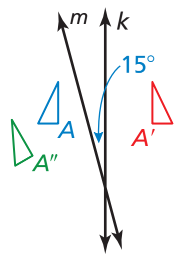
12. Is the distance from B' to m the same as the distance from B'' to m? Explain.

In Exercises 13 and 14, describe a single transformation that maps A to A''. *Example 4*

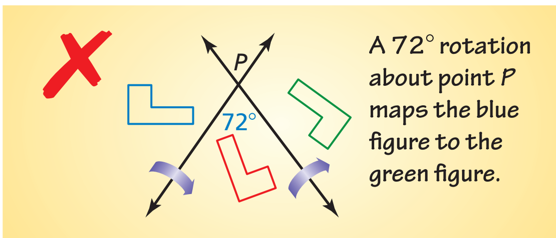
13.



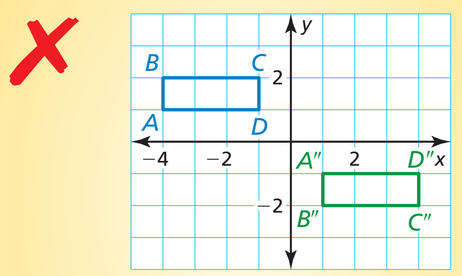
14.



15. **ERROR ANALYSIS** Describe and correct the error in using the Reflections in Intersecting Lines Theorem.



16. **ERROR ANALYSIS** Identify and correct the error in describing the congruence transformation.



Rectangle $ABCD$ is mapped to rectangle $A''B''C''D''$ by a translation 3 units down and a reflection in the y-axis.

In Exercises 17–20, find the measure of the acute or right angle formed by intersecting lines so that C can be mapped to C' using two reflections.

17. A rotation of 84° maps C to C'.

18. A rotation of 24° maps C to C'.

19. The rotation $(x, y) \to (-x, -y)$ maps C to C'.

20. The rotation $(x, y) \to (y, -x)$ maps C to C'.

21. MP **REASONING** Use the Reflections in Parallel Lines Theorem to explain how you can make a glide reflection using three reflections. How are the lines of reflection related?

22. MP **STRUCTURE** $\triangle ABC$ with vertices $A(1, 4)$, $B(4, 5)$, and $C(5, 1)$ is translated 6 units left and 2 units up, then rotated 180° about the origin. Determine which quadrant the image is located in without performing the transformations. Explain.

CRITICAL THINKING **In Exercises 23–26, tell whether the statement is *always*, *sometimes*, or *never* true. Explain your reasoning.**

23. A congruence transformation changes the size of a figure.

24. If two figures are congruent, then there is a rigid motion or a composition of rigid motions that maps one figure to the other.

25. The composition of two reflections results in a rotation.

26. The composition of two reflections results in a translation.

27. **MAKING AN ARGUMENT** $\overline{PQ}$ with endpoints $P(1, 3)$ and $Q(3, 2)$ is reflected in the y-axis. The image $\overline{P'Q'}$ is then reflected in the x-axis to produce the image $\overline{P''Q''}$. Your friend says that $\overline{PQ}$ is mapped to $\overline{P''Q''}$ by a 180° rotation about the origin. Is your friend correct? Explain your reasoning.

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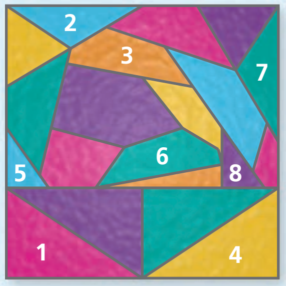
28. **HOW DO YOU SEE IT?** What type of congruence transformation verifies each statement about the stained glass window?



a. Triangle 5 is congruent to Triangle 8.

b. Triangle 1 is congruent to Triangle 4.

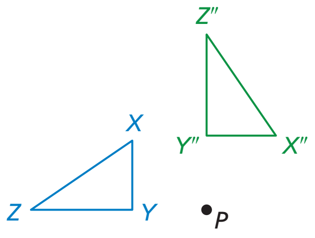
c. Triangle 2 is congruent to Triangle 7.

d. Pentagon 3 is congruent to Pentagon 6.

CONSTRUCTION **In Exercises 29 and 30, use a compass and straightedge to construct two lines of reflection that produce a composition of reflections resulting in the given transformation.**

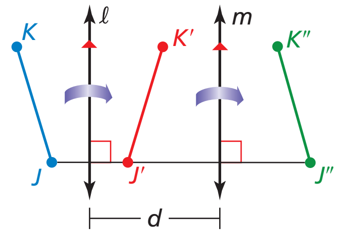
29. **Translation:** $\triangle ABC \to \triangle A''B''C''$

30. **Rotation about P:** $\triangle XYZ \to \triangle X''Y''Z''$



31. **OPEN-ENDED** A *tessellation* is the covering of a plane with one or more congruent figures so that there are no gaps or overlaps. An example of a tessellation is shown. Draw a tessellation that involves two or more different transformations. Describe the transformations.

32. PROVING A THEOREM Prove the Reflections in Parallel Lines Theorem.



Given A reflection in line ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in line m maps $\overline{J'K'}$ to $\overline{J''K''}$, and $\ell \parallel m$.

Prove **a.** $\overline{JJ''}$ is perpendicular to ℓ and m.

b. $JJ'' = 2d$, where d is the distance between ℓ and m.

33. CRITICAL THINKING You reflect a figure in each of two parallel lines. Is the order of the reflections important? Justify your answer.

GO DIGITAL

34. THOUGHT PROVOKING
Describe a sequence of two or more different transformations where the order of the transformations does not affect the final image.

35. DIG DEEPER Are any two rays congruent? If so, describe a congruence transformation that maps a ray to any other ray. If not, explain why not.

REVIEW & REFRESH

WATCH

In Exercises 36–38, solve the equation.

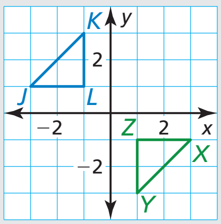
36. $12 + 6m = 2m$

37. $-2(8 - y) = -6y$

38. $7(2n + 1) = \frac{1}{3}(6n - 15)$

39. MODELING REAL LIFE Last month, a charity received \$500 in donations. This month, the charity received \$625 in donations. What is the percent of change?

40. Describe a congruence transformation that maps $\triangle JKL$ to $\triangle XYZ$.



In Exercises 41 and 42, graph quadrilateral *QRST* with vertices *Q*(2, −1), *R*(5, −2), *S*(5, −4), and *T*(2, −4) and its image after the composition.

41. Translation: $(x, y) \rightarrow (x - 5, y + 3)$
Reflection: in the line $y = -3$

42. Reflection: in the x-axis
Rotation: 90° about the origin

In Exercises 43 and 44, graph the linear equation. Identify the *x*-intercept.

43. $y = -\frac{3}{4}x + 2$

44. $3x + y = -5$

45. Write an inequality that represents the graph.

46. Let p be "you ride a roller coaster" and let q be "you go to an amusement park." Write the conditional statement $p \rightarrow q$, the converse $q \rightarrow p$, the inverse $\sim p \rightarrow \sim q$, and the contrapositive $\sim q \rightarrow \sim p$ in words. Then decide whether each statement is *true* or *false*.

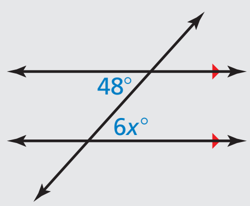
In Exercises 47 and 48, determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

47. $\frac{2}{3}, 2, 6, 18, \ldots$

48. $-4, -1, 2, 5, \ldots$

In Exercises 49 and 50, find the value of *x*. Show your steps.

49.



50.