2.4 Modeling with Quadratic Functions

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Learning Target Write equations of quadratic functions using given characteristics.

Success Criteria

- I can write equations of quadratic functions using vertices, points, and x-intercepts.
- I can write quadratic equations to model data sets.
- I can use technology to find a quadratic model for a set of data.

EXPLORE IT! Modeling with Quadratic Functions

Math Practice

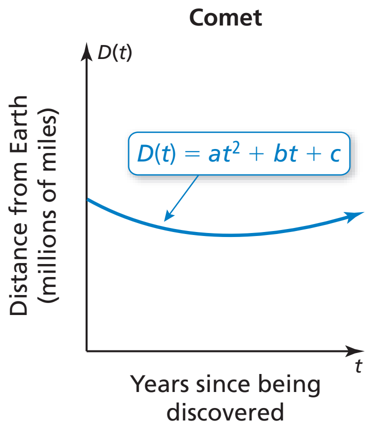
Use Other Resources How can you check your predictions when the values of a, b, or c change in the quadratic model?

Work with a partner.

a. Explain what the graph represents.

b. What do you know about the value of a? How does the graph change if a is increased? decreased? What does this mean in this context? Explain your reasoning.

c. Write an expression that represents the year t when the comet is closest to Earth.



d. The comet is the same distance away from Earth in 2012 and 2020. Estimate the year when the comet is closest to Earth. Explain your reasoning.

e. What does c represent in this context? How does the graph change if c is increased? decreased? Explain.

f. Assume that the model is still valid today. Is the comet's distance from Earth currently increasing, decreasing, or constant? Explain.

g. The table shows the approximate distances y (in millions of miles) from Earth for a planetary object m months after being discovered. Can you use a quadratic function to model the data? How do you know? Is this the only type of function you can use to model the data? Explain your reasoning.

Months, m	0	1	2	3	4	5	6	7	8	9
Distance (millions of miles), y	50	57	65	75	86	101	115	130	156	175

h. Explain how you can find a quadratic model for the data. How do you know your model is a good fit?

Writing Quadratic Equations

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KEY IDEA

Writing Quadratic Equations

Given a point and the vertex (h, k) — Use vertex form: $y = a(x - h)^2 + k$

Given a point and the x-intercepts p and q — Use intercept form: $y = a(x - p)(x - q)$

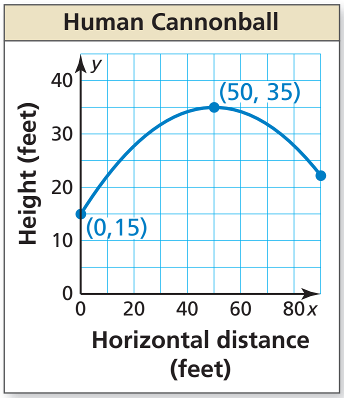
Given three points — Write and solve a system of three equations in three variables.

The world record for the farthest distance traveled by a human cannonball is almost 200 feet, achieved by David Smith Jr.

EXAMPLE 1 Writing an Equation Using the Vertex and a Point

WATCH

The graph shows the parabolic path of a performer who is shot out of a cannon, where y is the height (in feet) and x is the horizontal distance traveled (in feet). The performer lands in a net 90 feet from the cannon. What is the height of the net?



SOLUTION

From the graph, you can see that the vertex (h, k) is $(50, 35)$ and the parabola passes through the point $(0, 15)$. Use the vertex and the point to solve for a in vertex form. Then write an equation of the parabola.

$$y = a(x - h)^2 + k \qquad \text{Vertex form}$$

$$15 = a(0 - 50)^2 + 35 \qquad \text{Substitute for } h, k, x, \text{ and } y.$$

$$-20 = 2500a \qquad \text{Simplify.}$$

$$-0.008 = a \qquad \text{Divide each side by 2500.}$$

Because $a = -0.008$, $h = 50$, and $k = 35$, the path can be modeled by the equation $y = -0.008(x - 50)^2 + 35$, where $0 \le x \le 90$.

Find the height when $x = 90$.

$$y = -0.008(90 - 50)^2 + 35 \qquad \text{Substitute 90 for } x.$$

$$= -0.008(1600) + 35 \qquad \text{Simplify.}$$

$$= 22.2 \qquad \text{Simplify.}$$

▶ So, the height of the net is about 22 feet.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. **WHAT IF?** The vertex of the parabola is $(50, 37.5)$. What is the height of the net?

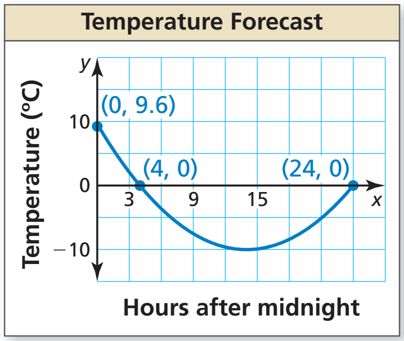
Write an equation of the parabola in vertex form.

2. passes through $(1, -7)$ and has vertex $(-2, 5)$
3. passes through $(0, 8)$ and has vertex $(-10, -3)$

EXAMPLE 2 Writing an Equation Using a Point and x-Intercepts

WATCH

GO DIGITAL



A meteorologist creates a parabola to predict the temperature tomorrow, where x is the number of hours after midnight and y is the temperature (in degrees Celsius).

a. Write a function f that models the temperature over time. What is the coldest temperature?

b. What is the average rate of change in temperature over the interval in which the temperature is decreasing? increasing? Compare the average rates of change.

SOLUTION

a. The x-intercepts are 4 and 24, and the parabola passes through (0, 9.6). Use the x-intercepts and the point to solve for a in intercept form.

$$y = a(x - p)(x - q) \qquad \text{Intercept form}$$

$$9.6 = a(0 - 4)(0 - 24) \qquad \text{Substitute for } p, q, x, \text{ and } y.$$

$$9.6 = 96a \qquad \text{Simplify.}$$

$$0.1 = a \qquad \text{Divide each side by 96.}$$

Because $a = 0.1$, $p = 4$, and $q = 24$, the temperature over time can be modeled by $f(x) = 0.1(x - 4)(x - 24)$, where $0 \le x \le 24$.

The coldest temperature is the minimum value. Find $f(x)$ when $x = \frac{4 + 24}{2} = 14$.

$$f(14) = 0.1(14 - 4)(14 - 24) \qquad \text{Substitute 14 for } x.$$

$$= -10 \qquad \text{Simplify.}$$

▶ So, the coldest temperature is -10°C at 14 hours after midnight, or 2 P.M.

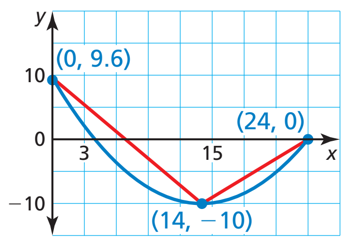
b. The parabola opens up and the axis of symmetry is $x = 14$. So, the function is decreasing over the interval $0 < x < 14$ and increasing over the interval $14 < x < 24$.

Average rate of change over $0 < x < 14$:

$$\frac{f(14) - f(0)}{14 - 0} = \frac{-10 - 9.6}{14} = -1.4$$

Average rate of change over $14 < x < 24$:

$$\frac{f(24) - f(14)}{24 - 14} = \frac{0 - (-10)}{10} = 1$$



▶ Because $|-1.4| > |1|$, the average rate at which the temperature decreases from midnight to 2 P.M. is greater than the average rate at which it increases from 2 P.M. to midnight.

REMEMBER

The average rate of change of a function f from x_1 to x_2 is the slope of the line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

4. **WHAT IF?** The y-intercept is 4.8. How does this change your answers in parts (a) and (b)?

5. **MP REASONING** In Example 2, compare the average rates of change over the intervals in which the temperature is below zero and decreasing, and below zero and increasing.

6. Write an equation of the parabola that passes through the point (2, 5) and has x-intercepts -2 and 4.

Writing Equations to Model Data

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When data have equally-spaced inputs, you can analyze patterns in the differences of the outputs to determine what type of function can be used to model the data. Linear data have constant *first differences*. Quadratic data have constant *second differences*. The first and second differences of $f(x) = x^2$ are shown below.

Equally-spaced x-values

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

first differences: -5 -3 -1 1 3 5

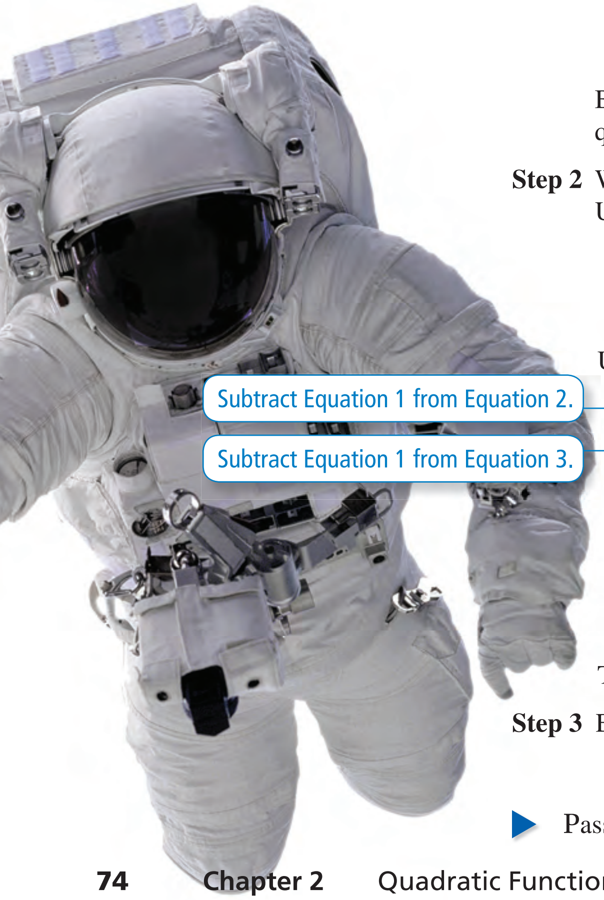
second differences: 2 2 2 2 2

EXAMPLE 3 **Writing a Quadratic Equation Using Three Points**

WATCH INFO

NASA can create a weightless environment by flying a plane in parabolic paths. The table shows the heights $h(t)$ (in feet) of a plane t seconds after starting the flight path. After about 20.8 seconds, passengers begin to experience a weightless environment. Write and evaluate a function to approximate the height at which this occurs.

Time, t	Height, $h(t)$
10	26,900
15	29,025
20	30,600
25	31,625
30	32,100
35	32,025
40	31,400



SOLUTION

Step 1 The input values are equally spaced. So, analyze the differences in the outputs to determine what type of function you can use to model the data.

$h(10)$	$h(15)$	$h(20)$	$h(25)$	$h(30)$	$h(35)$	$h(40)$
26,900	29,025	30,600	31,625	32,100	32,025	31,400

first differences: 2125 1575 1025 475 -75 -625

second differences: -550 -550 -550 -550 -550

Because the second differences are constant, you can model the data with a quadratic function.

Step 2 Write a quadratic function of the form $h(t) = at^2 + bt + c$ that models the data. Use any three points $(t, h(t))$ from the table to write a system of equations.

Use (10, 26,900): $100a + 10b + c = 26{,}900$ Equation 1
Use (20, 30,600): $400a + 20b + c = 30{,}600$ Equation 2
Use (30, 32,100): $900a + 30b + c = 32{,}100$ Equation 3

Use the elimination method to solve the system.

Subtract Equation 1 from Equation 2. → $300a + 10b = 3700$ New Equation 1

Subtract Equation 1 from Equation 3. → $800a + 20b = 5200$ New Equation 2

$200a = -2200$ Subtract 2 times new Equation 1 from new Equation 2.

$a = -11$ Solve for a.

$b = 700$ Substitute into new Equation 1 to find b.

$c = 21{,}000$ Substitute into Equation 1 to find c.

The data can be modeled by the function $h(t) = -11t^2 + 700t + 21{,}000$.

Step 3 Evaluate the function when $t = 20.8$.

$$h(20.8) = -11(20.8)^2 + 700(20.8) + 21{,}000 = 30{,}800.96$$

▶ Passengers begin to experience a weightless environment at about 30,800 feet.

Real-life data that show a quadratic relationship usually do not have constant second differences because the data are not *exactly* quadratic. Relationships that are *approximately* quadratic have second differences that are relatively "close" in value. Many technology tools have a *quadratic regression* feature that you can use to find a quadratic function that best models a set of data.

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EXAMPLE 4 Using Quadratic Regression

WATCH

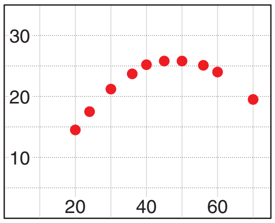
The table shows fuel efficiencies of a vehicle at different speeds. Write a function that models the data. Use the model to approximate the best gas mileage.

Miles per hour, x	Miles per gallon, y
20	14.5
24	17.5
30	21.2
36	23.7
40	25.2
45	25.8
50	25.8
56	25.1
60	24.0
70	19.5

SOLUTION

Because the x-values are not equally spaced, you cannot analyze the differences in the outputs. Use technology to find a function that models the data.

Step 1 Enter the data from the table and create a scatter plot. The data show a quadratic relationship.



Step 2 Find the quadratic equation. The values in the equation can be rounded to obtain $y = -0.014x^2 + 1.37x - 7.1$.

$y = ax^2 + bx + c$

PARAMETERS

$a = -0.0141$ $b = 1.3662$

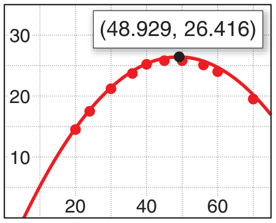
$c = -7.1441$

STATISTICS

$R^2 = 0.9992$

Step 3 Graph the regression equation with the scatter plot.

In this context, the best gas mileage is the maximum mileage per gallon. Using technology, you can see that the maximum mileage per gallon is about 26.4 miles per gallon when driving about 48.9 miles per hour.



So, the best gas mileage is about 26.4 miles per gallon.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

7. WRITING Explain when it is appropriate to use a quadratic model for a set of data.

8. Write an equation of the parabola that passes through the points $(-1, 4)$, $(0, 1)$, and $(2, 7)$.

9. The table shows the estimated profits y (in dollars) for a concert when the charge is x dollars per ticket. Write and evaluate a function to determine the maximum profit.

Ticket price, x	2	5	8	11	14	17
Profit, y	2600	6500	8600	8900	7400	4100

10. The table shows the results of an experiment testing the maximum weights y (in tons) supported by ice x inches thick. Write a function that models the data. How much weight can be supported by ice that is 22 inches thick?

Ice thickness, x	12	14	15	18	20	24	27
Maximum weight, y	3.4	7.6	10.0	18.3	25.0	40.6	54.3

2.4 Practice WITH CalcChat® AND CalcView®

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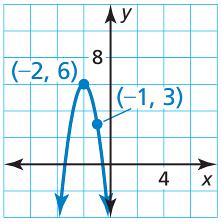
In Exercises 1–6, write an equation of the parabola in vertex form. *Example 1*

1.



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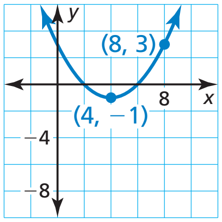


3. passes through (13, 8) and has vertex (3, 2)

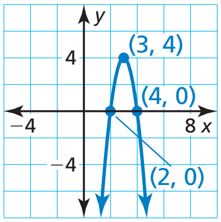
4. passes through (−7, −15) and has vertex (−5, 9)

5. passes through (0, −24) and has vertex (−6, −12)

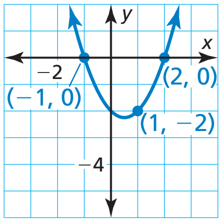
6. passes through (6, 35) and has vertex (−1, 14)

In Exercises 7–12, write an equation of the parabola in intercept form. *Example 2*

7.



8.



9. x-intercepts: 12 and −6; passes through (14, 4)

10. x-intercepts: 9 and 1; passes through (0, −18)

11. x-intercepts: −16 and −2; passes through (−18, 72)

12. x-intercepts: −7 and −3; passes through (−2, 0.05)

13. **WRITING** Explain when to use intercept form and when to use vertex form when writing an equation of a parabola.

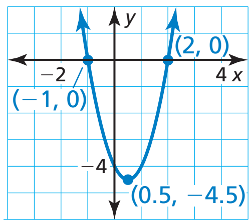
14. **COLLEGE PREP** Which of the following equations represent the parabola? Select all that apply.

Ⓐ $y = 2(x - 2)(x + 1)$

Ⓑ $y = 2(x + 0.5)^2 - 4.5$

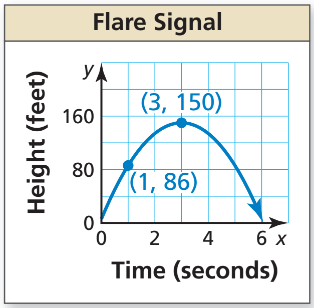
Ⓒ $y = 2(x - 0.5)^2 - 4.5$

Ⓓ $y = 2(x + 2)(x - 1)$

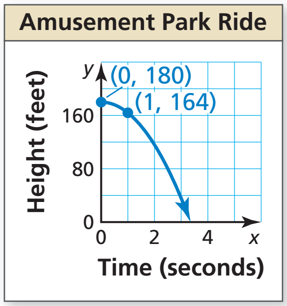


In Exercises 15–18, write an equation of the parabola in vertex form or intercept form.

15.



16.



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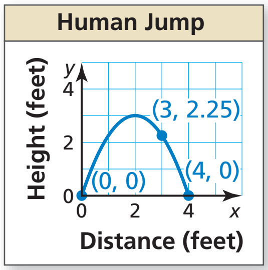
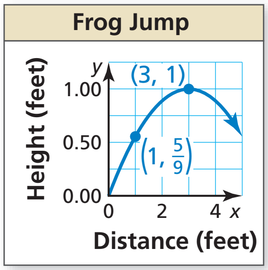
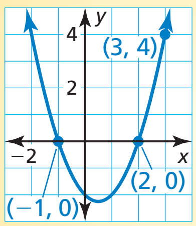


18.



19. **ERROR ANALYSIS** Describe and correct the error in writing an equation of the parabola.



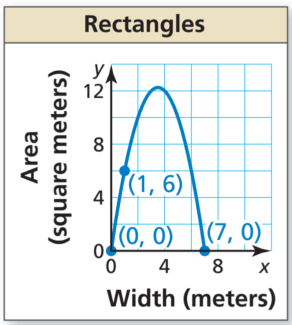
$y = a(x - p)(x - q)$

$4 = a(3 - 1)(3 + 2)$

$\frac{2}{5} = a$

$y = \frac{2}{5}(x - 1)(x + 2)$

20. **CONNECTING CONCEPTS** The area of a rectangle is modeled by the graph, where y is the area (in square meters) and x is the width (in meters). Write an equation of the parabola. Find the dimensions and corresponding area of one possible rectangle. What dimensions result in the maximum area?



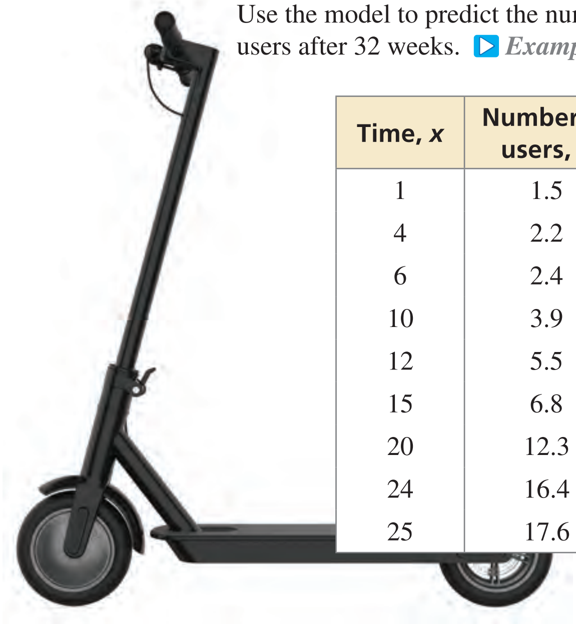
21. **MODELING REAL LIFE** Every rope has a safe working load. A rope should not be used to lift a weight greater than its safe working load. The table shows the safe working loads S (in pounds) for ropes with circumferences C (in inches). Write an equation for the safe working load for a rope. Find the safe working load for a rope that has a circumference of 10 inches. *Example 3*

Circumference, C	0	1	2	3
Safe working load, S	0	180	720	1620

22. **MODELING REAL LIFE** A baseball is thrown up in the air. The table shows the heights y (in feet) of the baseball after x seconds. Write an equation for the path of the baseball. Find the height of the baseball after 1.7 seconds.

Time, x	0.5	1	1.5	2
Baseball height, y	18	24	22	12

23. MP **USING TOOLS** The table shows the numbers y (in thousands) of people in a city who regularly use sharable electric scooters x weeks after the scooters are introduced. Write a function that models the data. Use the model to predict the number of users after 32 weeks. *Example 4*



Time, x	Number of users, y
1	1.5
4	2.2
6	2.4
10	3.9
12	5.5
15	6.8
20	12.3
24	16.4
25	17.6

24. MP **USING TOOLS** The table shows the numbers y of students absent from school x days after a flu outbreak. Write a function that models the data. Use the model to approximate the number of students absent 10 days after the outbreak.

Time (days), x	2	4	5	6	8	9	11
Number of students, y	11	17	19	19	17	14	7

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25. **COMPARING METHODS** Which method is more efficient for finding an equation of the parabola that passes through the points $(-8, 0)$, $(2, -20)$, and $(1, 0)$: using a system of three equations in three variables or using intercept form? Justify your answer.

26. **MAKING AN ARGUMENT** Do quadratic functions with the same x-intercepts have the same equations, vertex, and axis of symmetry? Explain your reasoning.

27. **MODELING REAL LIFE** The table shows the distances y a motorcyclist is from home after x hours.

Time (hours), x	0	1	2	3
Distance (miles), y	0	45	90	135

a. Determine what type of function you can use to model the data. Explain your reasoning.

b. Write and evaluate a function to determine the distance the motorcyclist is from home after 6 hours.

28. MP **PROBLEM SOLVING** The table shows the heights y of a competitive water-skier x seconds after jumping off a ramp. Write a function that models the height of the water-skier over time. When is the water-skier 5 feet above the water? How long is the skier in the air?

Time (seconds), x	0	0.25	0.75	1	1.1
Height (feet), y	22	22.5	17.5	12	9.24

In Exercises 29–32, analyze the differences in the outputs to determine whether the data are *linear*, *quadratic*, or *neither*. Explain. If the data are linear or quadratic, write a function that models the data.

29.

Price decrease (dollars), x	0	5	10	15	20
Revenue ($1000s), y	470	630	690	650	510

30.

Time (hours), x	0	1	2	3	4
Height (feet), y	40	42	44	46	48

31.

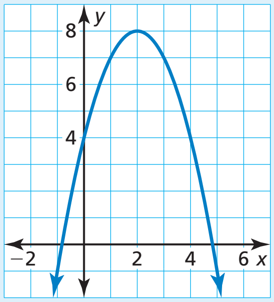
Time (hours), x	1	2	3	4	5
Population (hundreds), y	2	4	8	16	32

32.

Day, x	1	2	3	4
Balance (dollars), y	320	303	254	173

33. **OPEN-ENDED** Describe a real-life situation not mentioned in this chapter that can be modeled by a quadratic equation. Justify your answer.

34. **HOW DO YOU SEE IT?**
Use the graph to determine whether the average rate of change over each interval is *positive*, *negative*, or *zero*.



a. $0 \le x \le 2$ b. $2 \le x \le 5$

c. $2 \le x \le 4$ d. $0 \le x \le 4$

35. **REPEATED REASONING** The table shows the number of tiles in each figure. Verify that the data show a quadratic relationship. Predict the number of tiles in the 12th figure.

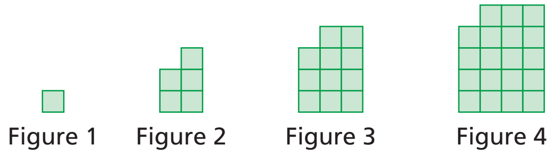


Figure	1	2	3	4
Number of tiles	1	5	11	19

36. **THOUGHT PROVOKING**
The table shows the temperatures y (in degrees Fahrenheit) of a cup of tea after x minutes. Write a function that models the data and can be used to predict the temperature of the tea after 20 minutes. Explain your reasoning.

Time, x	0	2	4	6	8	10
Temperature, y	190	164	146	131	120	111

REVIEW & REFRESH

In Exercises 37–40, factor the polynomial.

37. $x^2 + 4x + 3$

38. $x^2 - 3x + 2$

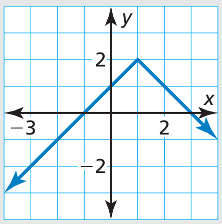
39. $3x^2 - 15x + 12$

40. $x^2 + x - 6$

41. **MODELING REAL LIFE** The table shows the heights y (in feet) of a firework x seconds after it is launched. The firework explodes at its highest point. Write an equation for the path of the firework. Find the height at which the firework explodes.

Time, x	0	1	2	3
Height, y	0	112	192	240

42. Determine whether the graph represents a function. Explain.



In Exercises 43 and 44, graph the function. Label the vertex and axis of symmetry.

43. $f(x) = 2(x - 1)^2 - 5$

44. $h(x) = 3x^2 + 6x - 2$

In Exercises 45 and 46, identify the focus, directrix, and axis of symmetry of the parabola. Graph the equation.

45. $x = -\frac{1}{12}y^2$

46. $16y = x^2$

47. Let the graph of g be a horizontal shrink by a factor of $\frac{1}{4}$, followed by a translation 1 unit up and 3 units right of the graph of $f(x) = (2x + 1)^2 - 11$. Write a rule for g and identify the vertex.

In Exercises 48–51, solve the inequality. Graph the solution.

48. $m + 9 \ge 13$

49. $15 - n < -6$

50. $5p > 10$

51. $-\frac{q}{4} \le 3$

52. Determine whether the table represents a *linear* or an *exponential* function. Explain.

x	−1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{2}$	2	8	32

In Exercises 53 and 54, write an equation in slope-intercept form of the line that passes through the given points.

53. $(4, -1), (0, 3)$

54. $(-3, -2), (1, 4)$