1.6 Solving Absolute Value Equations

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Learning Target Write and solve equations involving absolute value.

Success Criteria
- I can write the two linear equations related to a given absolute value equation.
- I can solve equations involving one or two absolute values.
- I can identify special solutions of absolute value equations.

EXPLORE IT! Solving an Absolute Value Equation

Work with a partner. Consider the absolute value equation

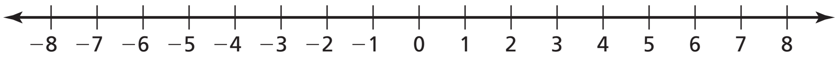
$$|x + 2| = 3.$$

a. Explain what you think this equation means.

b. Can you find a number that makes the equation true? If so, what is the number?

c. Do you think there is another number that makes the equation true? If so, find that number. Compare your answer with your classmates.

d. On the real number line below, locate the point for which the expression $|x + 2|$ is equal to 0.



Then locate the numbers you found in parts (b) and (c) on the real number line. What do you notice?

e. Complete the two linear equations below so that the solutions are the values you found in parts (b) and (c).

$$x + 2 = \underline{\qquad} \qquad x + 2 = \underline{\qquad}$$

f. Describe how to find the solutions of the absolute value equations algebraically. Then find the solutions.

i. $|x + 2| = 5$

ii. $|x - 3| = 1$

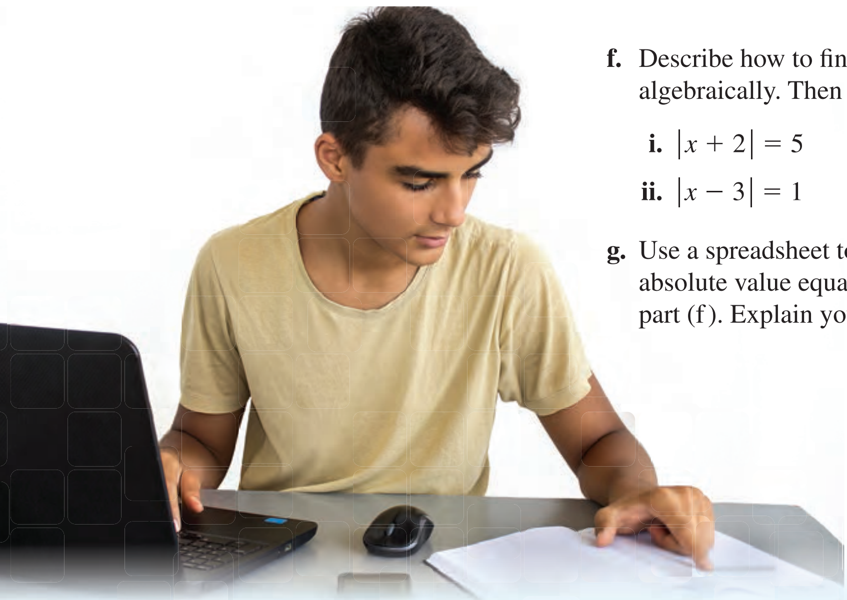
g. Use a spreadsheet to solve the absolute value equations in part (f). Explain your method.

	A	B
1	x	$\|x + 2\|$
2	−8	6
3	−7	
4	−6	
5	−5	
6	−4	
7	−3	
8	−2	
9	−1	
10	0	
11		

=abs(A2 + 2)

Math Practice

Construct Arguments
Construct a viable argument as to why you think there is or is not more than one solution to the absolute value equation.



Solving Absolute Value Equations

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Vocabulary

absolute value equation, p. 38
extraneous solution, p. 41

An **absolute value equation** is an equation that contains an absolute value expression. You can solve these types of equations by solving two related linear equations.

KEY IDEAS

Properties of Absolute Value

Let a and b be real numbers. Then the following properties are true.

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a||b|$ **4.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, b \neq 0$

Solving Absolute Value Equations

To solve $|ax + b| = c$ when $c \geq 0$, solve the related linear equations

$$ax + b = c \quad \text{or} \quad ax + b = -c.$$

When $c < 0$, the absolute value equation $|ax + b| = c$ has no solution because absolute value represents a distance and cannot be negative.

EXAMPLE 1 Solving Absolute Value Equations

WATCH

Solve each equation. Graph the solutions, if possible.

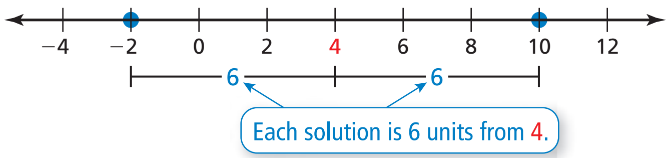
a. $|x - 4| = 6$ **b.** $|3x + 1| = -5$

SOLUTION

a. Write the two related linear equations for $|x - 4| = 6$. Then solve.

$x - 4 = 6$	*or*	$x - 4 = -6$	Write related linear equations.
$x = 10$		$x = -2$	Add 4 to each side.

▶ The solutions are $x = 10$ and $x = -2$.



b. The absolute value of an expression must be greater than or equal to 0. The expression $|3x + 1|$ cannot equal -5.

▶ So, the equation has no solution.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Graph the solutions, if possible.

1. $|x| = 10$ **2.** $|x - 1| = 4$ **3.** $|3 + x| = -\frac{1}{2}$

4. MP **REASONING** How do you know that the equation $|4x - 7| = -1$ has no solution?

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EXAMPLE 2 Solving a Multi-Step Absolute Value Equation

WATCH

Solve $|3x + 9| - 10 = -4$.

SOLUTION

ANOTHER WAY

Using the Product Property of Absolute Value, $|ab| = |a|\,|b|$, you can first rewrite the equation as

$$3|x + 3| - 10 = -4$$

and then solve.

First isolate the absolute value expression on one side of the equation.

$$|3x + 9| - 10 = -4 \qquad \text{Write the equation.}$$
$$|3x + 9| = 6 \qquad \text{Add 10 to each side.}$$

Now write the two related linear equations for $|3x + 9| = 6$. Then solve.

$$3x + 9 = 6 \quad \textit{or} \quad 3x + 9 = -6 \qquad \text{Write related linear equations.}$$
$$3x = -3 \qquad 3x = -15 \qquad \text{Subtract 9 from each side.}$$
$$x = -1 \qquad x = -5 \qquad \text{Divide each side by 3.}$$

▶ The solutions are $x = -1$ and $x = -5$.

EXAMPLE 3 Modeling Real Life

WATCH

You are driving on a highway and are about 250 miles from your state's border. You set your cruise control at 60 miles per hour and plan to turn it off within 30 miles of the border on either side. Find the minimum and maximum numbers of hours you will have cruise control on.

SOLUTION

One way to solve is to write an absolute value equation that models the number x of hours you will have cruise control on. You know that the distance you travel will be within 30 miles of 250 miles.

$$\underbrace{|\overset{\text{distance you travel}}{60x} - 250|}_{\text{distance from state border}} = \overset{\text{30 miles from the state border}}{30}$$

Write the two related linear equations for $|60x - 250| = 30$. Then solve.

$$60x - 250 = 30 \quad \textit{or} \quad 60x - 250 = -30 \qquad \text{Write related linear equations.}$$
$$60x = 280 \qquad 60x = 220 \qquad \text{Add 250 to each side.}$$
$$x = 4\tfrac{2}{3} \qquad x = 3\tfrac{2}{3} \qquad \text{Divide each side by 60.}$$

The solutions are $x = 3\frac{2}{3}$ and $x = 4\frac{2}{3}$.

Check

Minimum

$$60\left(3\tfrac{2}{3}\right) - 250 \stackrel{?}{=} -30$$
$$-30 = -30 \ \checkmark$$

Maximum

$$60\left(4\tfrac{2}{3}\right) - 250 \stackrel{?}{=} 30$$
$$30 = 30 \ \checkmark$$

▶ So, you will travel at least $3\frac{2}{3}$ hours and at most $4\frac{2}{3}$ hours with cruise control on.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Check your solutions.

5. $|x - 2| + 5 = 9$

6. $4|2x + 7| = 16$

7. $-2|5x - 1| - 3 = -11$

8. A plane is flying at a speed of 150 miles per hour. The pilot plans on flying at this speed for the next 160 miles, plus or minus 25 miles. Write an absolute value equation to find the minimum and maximum number of hours the plane will travel at that speed.

Solving Equations with Two Absolute Values

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If the absolute values of two algebraic expressions are equal, then they must either be equal to each other or be opposites of each other.

KEY IDEA

Solving Equations with Two Absolute Values

To solve $|ax + b| = |cx + d|$, solve the related linear equations

$$ax + b = cx + d \quad or \quad ax + b = -(cx + d).$$

EXAMPLE 4 **Solving Equations with Two Absolute Values** WATCH

Solve (a) $|3x - 4| = |x|$ and (b) $|4x - 10| = 2|3x + 1|$.

SOLUTION

Check

$$|3x - 4| = |x|$$
$$|3(2) - 4| \stackrel{?}{=} |2|$$
$$|2| \stackrel{?}{=} |2|$$
$$2 = 2 \checkmark$$

$$|3x - 4| = |x|$$
$$|3(1) - 4| \stackrel{?}{=} |1|$$
$$|-1| \stackrel{?}{=} |1|$$
$$1 = 1 \checkmark$$

a. Write the two related linear equations for $|3x - 4| = |x|$. Then solve.

$$\begin{array}{rclcrcl} 3x - 4 &=& x & \quad or \quad & 3x - 4 &=& -x \\ -x & & -x & & +x & & +x \\ 2x - 4 &=& 0 & & 4x - 4 &=& 0 \\ +4 & & +4 & & +4 & & +4 \\ 2x &=& 4 & & 4x &=& 4 \\ \frac{2x}{2} &=& \frac{4}{2} & & \frac{4x}{4} &=& \frac{4}{4} \\ x &=& 2 & & x &=& 1 \end{array}$$

▶ The solutions are $x = 2$ and $x = 1$.

b. Write the two related linear equations for $|4x - 10| = 2|3x + 1|$. Then solve.

$$\begin{array}{rclcrcl} 4x - 10 &=& 2(3x + 1) & \quad or \quad & 4x - 10 &=& 2[-(3x + 1)] \\ 4x - 10 &=& 6x + 2 & & 4x - 10 &=& 2(-3x - 1) \\ -6x & & -6x & & 4x - 10 &=& -6x - 2 \\ -2x - 10 &=& 2 & & +6x & & +6x \\ +10 & & +10 & & 10x - 10 &=& -2 \\ -2x &=& 12 & & +10 & & +10 \\ \frac{-2x}{-2} &=& \frac{12}{-2} & & 10x &=& 8 \\ x &=& -6 & & \frac{10x}{10} &=& \frac{8}{10} \\ & & & & x &=& 0.8 \end{array}$$

▶ The solutions are $x = -6$ and $x = 0.8$.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Check your solutions.

9. $|x + 8| = |2x + 1|$

10. $3|x - 4| = |2x + 5|$

11. $\frac{1}{2}|x + 8| = |4x - 1|$

Identifying Special Solutions

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When you solve an absolute value equation, it is possible for a solution to be *extraneous*. An **extraneous solution** is an apparent solution that must be rejected because it does not satisfy the original equation.

WORDS AND MATH

The word *extraneous* contains the prefix *extra-*, which means "more than is necessary." Extraneous information is information that is not relevant or essential.

EXAMPLE 5 Identifying Extraneous Solutions

WATCH

Solve $|2x + 12| = 4x$. Check your solutions.

SOLUTION

Write the two related linear equations for $|2x + 12| = 4x$. Then solve.

$2x + 12 = 4x$	*or*	$2x + 12 = -4x$	Write related linear equations.
$12 = 2x$		$12 = -6x$	Subtract $2x$ from each side.
$6 = x$		$-2 = x$	Solve for x.

Check

$$|2x + 12| = 4x$$
$$|2(6) + 12| \stackrel{?}{=} 4(6)$$
$$|24| \stackrel{?}{=} 24$$
$$24 = 24 \checkmark$$

$$|2x + 12| = 4x$$
$$|2(-2) + 12| \stackrel{?}{=} 4(-2)$$
$$|8| \stackrel{?}{=} -8$$
$$8 \neq -8 \times$$

Check the apparent solutions to see if either is extraneous.

▶ The solution is $x = 6$. Reject $x = -2$ because it is extraneous.

When solving equations of the form $|ax + b| = |cx + d|$, it is possible that one of the related linear equations will not have a solution.

EXAMPLE 6 Solving an Equation with Two Absolute Values

WATCH

Solve $|x + 5| = |x + 11|$.

SOLUTION

By equating the expression $x + 5$ and the opposite of $x + 11$, you obtain

$x + 5 = -(x + 11)$	Write related linear equation.
$x + 5 = -x - 11$	Distributive Property
$2x + 5 = -11$	Add x to each side.
$2x = -16$	Subtract 5 from each side.
$x = -8.$	Divide each side by 2.

However, by equating the expressions $x + 5$ and $x + 11$, you obtain

$x + 5 = x + 11$	Write related linear equation.
$x = x + 6$	Subtract 5 from each side.
$0 = 6$ ✗	Subtract x from each side.

which is a false statement. So, the original equation has only one solution.

▶ The solution is $x = -8$.

REMEMBER

Always check your solutions in the original equation to make sure they are not extraneous.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Check your solutions.

12. $|x + 6| = 2x$

13. $|3x - 2| = x$

14. $|2 + x| = |x - 8|$

15. $|5x - 2| = |5x + 4|$

16. **WRITING** How is solving an absolute value equation similar to solving an equation without an absolute value? How is it different?

1.6 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, simplify the expression.

1. $|-9|$

2. $-|15|$

3. $|14| - |-14|$

4. $|-3| + |3|$

5. $-|-5 \cdot (-7)|$

6. $|-0.8 \cdot 10|$

7. $\left|\dfrac{27}{-3}\right|$

8. $\left|-\dfrac{-12}{4}\right|$

In Exercises 9–22, solve the equation. Graph the solution(s), if possible. *Examples 1 and 2*

9. $|r| = -2$

10. $|x| = 13.4$

11. $|m + 3| = 7$

12. $|q - 8| = 14$

13. $\left|\dfrac{t}{2}\right| = 6$

14. $|-3.5d| = 15.4$

15. $|4b - 5| = 19$

16. $|x - 1| + 5 = 2$

17. $2|-8w + 6| = 76$

18. $\left|\dfrac{1}{3}y - 2\right| - 7 = 3$

19. $-4|8 - 5n| = 13$

20. $-3\left|1 - \dfrac{2}{3}v\right| = -9$

21. $3 = -2\left|\dfrac{1}{4}s - 5\right| + 3$

22. $9|4p + 2| + 8 = 35$

23. MODELING REAL LIFE The average distance from Earth to the Sun is 92.95 million miles. The actual distance varies from the average by up to 1.55 million miles. Write and solve an absolute value equation to find the minimum and maximum distance from Earth to the Sun. *Example 3*

24. MODELING REAL LIFE The recommended mass of a soccer ball is 0.43 kilogram. The actual mass is allowed to vary by up to 20 grams.

a. Write and solve an absolute value equation to find the minimum and maximum acceptable soccer ball masses.

b. A soccer ball has a mass of 423 grams. The soccer ball loses 0.016 kilogram of mass over time. Is the mass now acceptable? Explain.

MP STRUCTURE In Exercises 25–28, match the absolute value equation with its graph without solving the equation.

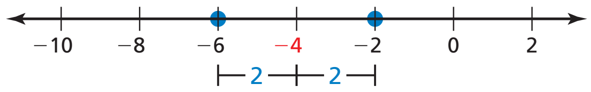
25. $|x + 2| = 4$

26. $|x - 4| = 2$

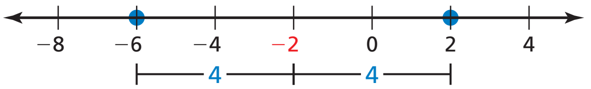
27. $|x - 2| = 4$

28. $|x + 4| = 2$

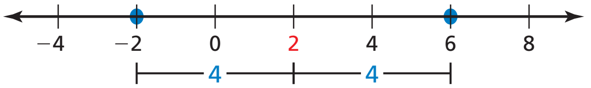
A.



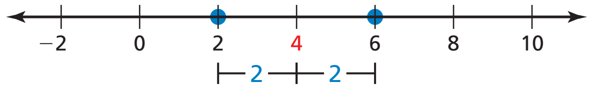
B.



C.



D.



In Exercises 29–38, solve the equation. Check your solutions. *Examples 4, 5, and 6*

29. $|4n - 15| = |n|$

30. $|2c + 8| = |10c|$

31. $|3k - 2| = 2|k + 2|$

32. $\left|\frac{1}{2}b - 8\right| = \left|\frac{1}{4}b - 1\right|$

33. $4|p - 3| = |2p + 8|$

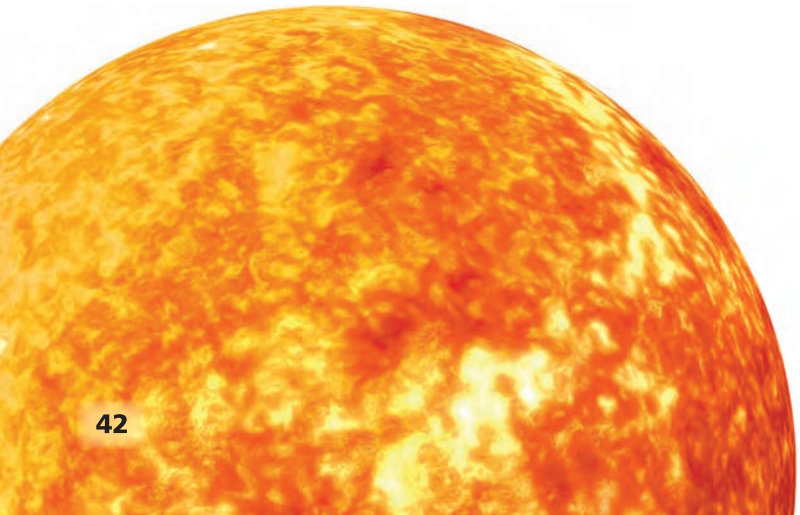
34. $2|4w - 1| = 3|4w + 2|$

35. $|3h + 1| = 7h$

36. $|6a - 5| = 4a$

37. $\left|f - \frac{4}{3}\right| = \left|f + \frac{1}{6}\right|$

38. $|3x - 4| = |3x - 5|$



MP REASONING **In Exercises 39–42, write an absolute value equation that has the given solutions.**

39. $x = 8$ and $x = 18$ **40.** $x = -6$ and $x = 10$

41. $x = 1.5$ and $x = 8.5$ **42.** $x = -10$ and $x = -5$

ERROR ANALYSIS **In Exercises 43 and 44, describe and correct the error in solving the equation.**

43.

$$|2x - 1| = -9$$
$$2x - 1 = -9 \quad or \quad 2x - 1 = -(-9)$$
$$2x = -8 \qquad 2x = 10$$
$$x = -4 \qquad x = 5$$

The solutions are $x = -4$ and $x = 5$.

44.

$$|5x + 8| = x$$
$$5x + 8 = x \quad or \quad 5x + 8 = -x$$
$$4x + 8 = 0 \qquad 6x + 8 = 0$$
$$4x = -8 \qquad 6x = -8$$
$$x = -2 \qquad x = -\frac{4}{3}$$

The solutions are $x = -2$ and $x = -\frac{4}{3}$.

45. MODELING REAL LIFE Starting from 300 feet away, a car drives toward you. It then passes by you at a constant speed of 48 feet per second. The distance d (in feet) of the car from you after t seconds is given by the equation $d = |300 - 48t|$.

a. Explain what $48t$ represents in the given equation.

b. At what times is the car 60 feet from you?

46. MP REASONING Without solving completely, place each equation into one of the three categories. Explain your reasoning.

No solution	One solution	Two solutions

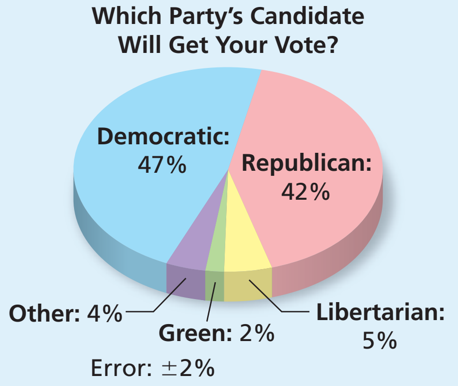
$|x - 2| + 6 = 0$ $|x + 3| - 1 = 0$

$|x + 8| + 2 = 7$ $|x - 1| + 4 = 4$

$|x - 6| - 5 = -9$ $|x + 5| - 8 = -8$

47. MAKING AN ARGUMENT Your friend says that the absolute value equation $|3x + 8| - 9 = -5$ has no solution because the constant on the right side of the equation is negative. Is your friend correct? Explain.

48. HOW DO YOU SEE IT?
The circle graph shows the results of a survey of registered voters the day of an election.



The error given in the graph means that the actual percent could be 2% more or 2% less than the percent reported by the survey.

a. What does the survey predict are the minimum and maximum percents of voters who will vote Republican? Green?

b. Write absolute value equations to represent your answers in part (a).

c. One candidate receives 44% of the vote. Which party do you think the candidate belongs to? Explain.

ABSTRACT REASONING **In Exercises 49–52, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.**

49. If $x^2 = a^2$, then $|x|$ is ________ equal to $|a|$.

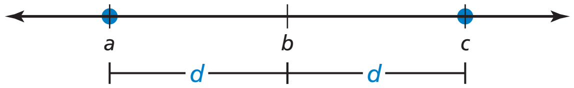
50. If a and b are real numbers, then $|a - b|$ is ________ equal to $|b - a|$.

51. For any real number p, the equation $|x - 4| = p$ will ________ have two solutions.

52. For any real number p, the equation $|x - p| = 4$ will ________ have two solutions.

53. WRITING Explain why absolute value equations can have no solution, one solution, or two solutions. Give an example of each case.

54. **MP STRUCTURE** Complete the equation $|x - \square| = \square$ with a, b, c, or d so that the equation is graphed correctly.



55. **COLLEGE PREP** Which values are solutions of the equation $5 = -\frac{2}{3}|4x - 7| + 11$? Select all that apply.

Ⓐ $x = -\frac{1}{2}$ Ⓑ $x = \frac{1}{2}$

Ⓒ $x = \frac{3}{4}$ Ⓓ $x = \frac{11}{4}$

Ⓔ $x = 4$ Ⓕ no solution

56. **CRITICAL THINKING** Solve the equation shown. Explain how you found your solution(s).

$$8|x + 2| - 6 = 5|x + 2| + 3$$

57. **OPEN-ENDED** Describe a real-life situation that can be modeled by an absolute value equation with the solutions $x = 62$ and $x = 72$.

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58. **THOUGHT PROVOKING**

What is the maximum number of solutions an equation of the form $\big||ax - b| + c\big| = d$ can have? Justify your reasoning with an example.

59. **DIG DEEPER** The minimum normal glucose level for a fasting adult is 70 mg/dL. The maximum normal level is 99 mg/dL. Write an absolute value equation that represents the minimum and maximum normal glucose levels.

60. **ABSTRACT REASONING** How many solutions does the equation $a|x + b| + c = d$ have when $a > 0$ and $c = d$? when $a < 0$ and $c > d$? Explain your reasoning.

REVIEW & REFRESH

WATCH

In Exercises 61–64, solve the equation. Check your solution(s).

61. $3c + 1 = c + 1$

62. $4(6k + 9) = 8(3k - 2)$

63. $-10 - 12g = -4(3g + 2.5)$

64. $|y - 4| = |y + 10|$

65. **MODELING REAL LIFE** An outdoor music festival provides 4000 square yards of land for the audience. Attendees are permitted to reserve a section using a rectangular tarp with a length of 12 feet and a width of 10 feet. About how many sections can be reserved at the music festival?

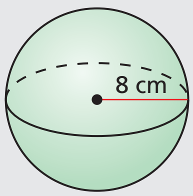
66. Simplify $12 + 5h - 3.5 + 8h$.

In Exercises 67 and 68, write the number in standard form.

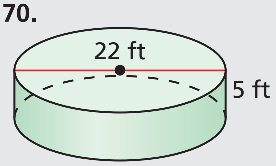
67. 7×10^{-8} **68.** 2.59×10^{3}

In Exercises 69 and 70, find the volume of the figure. Round your answer to the nearest tenth.

69.



70.



JUSTIFYING STEPS In Exercises 71 and 72, identify the property of equality that can be used to justify that Equation 1 and Equation 2 are equivalent.

71.

Equation 1	$3x + 8 = x - 1$
Equation 2	$3x + 9 = x$

72.

Equation 1	$4y = 28$
Equation 2	$y = 7$

73. A circle has an area of 36π square inches. Find the radius.

74. A triangle has a height of 8 feet and an area of 48 square feet. Find the base.

75. **MODELING REAL LIFE** You are driving your moped to school. The drive is about 12.5 miles, but the distance varies by up to 1.25 miles, depending on the route you take. You drive at a constant speed of 25 miles per hour. Find the minimum and maximum number of minutes it will take you to travel to school.

In Exercises 76–79, complete the statement. Round to the nearest hundredth, if necessary.

76. 9900 sec = $\square$ h **77.** 0.25 T = $\square$ oz

78. 11.5 qt ≈ $\square$ mL **79.** 49.6 cm ≈ $\square$ ft